15. INTEGRATION USING PARTIAL FRACTIONS

Partial Fractions

RATIONAL FUNCTIONS *If* $f(x)$ and $g(x)$ *are polynomial functions such that* $g(x) \neq 0$ *then* $\dfrac{f(x)}{g(x)}$ *is called a rational function.*

If degree $f(x) <$ degree $g(x)$ then $\dfrac{f(x)}{g(x)}$ is called a *proper rational function.*

If degree $f(x) \geq$ degree $g(x)$ then $\dfrac{f(x)}{g(x)}$ is called an *improper rational function.*

If $\dfrac{f(x)}{g(x)}$ is an improper rational function then by dividing $f(x)$ by $g(x)$, we can express $\dfrac{f(x)}{g(x)}$ as the sum of a polynomial and a proper rational function.

PARTIAL FRACTIONS *Any proper rational function* $\dfrac{p(x)}{q(x)}$ *can be expressed as the sum of rational functions, each having a simplest factor* $q(x)$. Each such fraction is known as a partial fraction and the process of obtaining them is called the decomposition or resolving of the given function into partial fractions.

METHOD We first resolve the denominator of the given fraction into simplest factors. On the basis of these factors, we obtain the corresponding partial fraction as per rules given below:

Factor in the denominator	Corresponding partial fraction
(i) $(x-a)$	$\dfrac{A}{(x-a)}$
(ii) $(x-b)^2$	$\dfrac{A}{(x-b)} + \dfrac{B}{(x-b)^2}$
(iii) $(x-c)^3$	$\dfrac{A}{(x-c)} + \dfrac{B}{(x-c)^2} + \dfrac{C}{(x-c)^3}$
(iv) (ax^2+bx+c)	$\dfrac{Ax+B}{(ax^2+bx+c)}$

The values of A, B, C, etc., can be obtained as shown below.

SOLVED EXAMPLES ON PARTIAL FRACTIONS

EXAMPLE 1 *Resolve* $\dfrac{2x+3}{(x-3)(x+1)}$ *into partial fractions.*

SOLUTION Let $\dfrac{(2x+3)}{(x-3)(x+1)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x+1)}$. Then,

$$\frac{2x+3}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

or $(2x+3) \equiv A(x+1)+B(x-3)$. … (i)

Putting $(x-3)=0$ or $x=3$ in (i), we get $A=(9/4)$.

Putting $(x+1)=0$ or $x=-1$ in (i), we get $B=(-1/4)$.

$$\therefore \quad \frac{(2x+3)}{(x-3)(x+1)} = \frac{9}{4(x-3)} - \frac{1}{4(x+1)}.$$

EXAMPLE 2 *Resolve* $\dfrac{x^3-2x^2-13x-12}{x^2-3x-10}$ *into partial fractions.*

SOLUTION On dividing, we get

$$\frac{x^3-2x^2-13x-12}{x^2-3x-10} = (x+1) - \frac{2}{(x^2-3x-10)}. \qquad \text{… (i)}$$

Let $$\frac{2}{(x^2-3x-10)} = \frac{2}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$$

Then, $$\frac{2}{(x-5)(x+2)} = \frac{A(x+2)+B(x-5)}{(x-5)(x+2)}$$

or $2 \equiv A(x+2)+B(x-5)$. … (ii)

Putting $(x-5)=0$ or $x=5$ in (ii), we get $A=(2/7)$.

Putting $(x+2)=0$ or $x=-2$ in (ii), we get $B=(-2/7)$.

$$\therefore \quad \frac{2}{(x^2-3x-10)} = \frac{2}{7(x-5)} - \frac{2}{7(x+2)}.$$

Hence, $$\frac{x^3-2x^2-13x-12}{x^2-3x-10} = (x+1) - \frac{2}{7(x-5)} + \frac{2}{7(x+2)}.$$

EXAMPLE 3 *Resolve* $\dfrac{16}{(x-2)(x+2)^2}$ *into partial fractions.*

SOLUTION Let $$\frac{16}{(x-2)(x+2)^2} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

or $$\frac{16}{(x-2)(x+2)^2} = \frac{A(x+2)^2 + B(x-2)(x+2) + C(x-2)}{(x-2)(x+2)^2}$$

$\therefore \quad 16 \equiv A(x+2)^2 + B(x-2)(x+2) + C(x-2) \quad \ldots$ (i)

or $\quad 16 \equiv (A+B)x^2 + (4A+C)x + (4A-4B-2C). \quad \ldots$ (ii)

Putting $(x-2)=0$ or $x=2$ in (i), we get $A=1$.

Putting $(x+2)=0$ or $x=-2$ in (i), we get $C=-4$.

Comparing the coefficients of x^2 on both sides of (ii), we get

$$A+B=0 \text{ or } B=-A=-1.$$

Thus $A=1,\ B=-1$ and $C=-4$.

$$\therefore \quad \frac{16}{(x-2)(x+2)^2} = \left[\frac{1}{(x-2)} - \frac{1}{(x+2)} - \frac{4}{(x+2)^2}\right].$$

EXAMPLE 4 *Resolve* $\dfrac{2x+1}{(x-1)(x^2+1)}$ *into partial fractions.*

SOLUTION Let $\dfrac{2x+1}{(x-1)(x^2+1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+1)}$

or $\dfrac{(2x+1)}{(x-1)(x^2+1)} = \dfrac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}.$

$\therefore \quad 2x+1 \equiv A(x^2+1)+(Bx+C)(x-1)$

or $\quad 2x+1 \equiv (A+B)x^2 + (C-B)x + (A-C). \quad \ldots$ (i)

Equating the like powers of x on both sides of (i), we get

$$A+B=0,\ C-B=2 \text{ and } A-C=1.$$

On solving these equations, we get

$$A=\frac{3}{2},\ B=\frac{-3}{2} \text{ and } C=\frac{1}{2}.$$

$$\therefore \quad \frac{2x+1}{(x-1)(x^2+1)} = \frac{3}{2(x-1)} + \frac{\left(\frac{-3}{2}x+\frac{1}{2}\right)}{x^2+1} = \left[\frac{3}{2(x-1)} + \frac{(1-3x)}{2(x^2+1)}\right].$$

Integration Using Partial Fractions

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\displaystyle\int \frac{(x-1)}{(x+1)(x-2)}dx$. **[CBSE 2001]**

SOLUTION Let $\dfrac{(x-1)}{(x+1)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}.$

Then, $(x-1) \equiv A(x-2) + B(x+1). \quad \ldots$ (i)

Putting $x=-1$ in (i), we get $A=\dfrac{2}{3}$.

Putting $x=2$ in (i), we get $B=\dfrac{1}{3}$.

$$\therefore \quad \frac{(x-1)}{(x+1)(x-2)} = \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

$$\Rightarrow \int \frac{(x-1)}{(x+1)(x-2)}dx = \frac{2}{3}\int \frac{dx}{(x+1)} + \frac{1}{3}\int \frac{dx}{(x-2)}$$

$$= \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C.$$

EXAMPLE 2 *Evaluate* $\int \frac{(x^2+1)}{(x^2-5x+6)}dx.$

SOLUTION Here the integrand is not a proper rational function; on dividing (x^2+1) by (x^2-5x+6), we get

$$\frac{(x^2+1)}{(x^2-5x+6)} = 1 + \frac{(5x-5)}{(x^2-5x+6)} = 1 + \frac{(5x-5)}{(x-2)(x-3)}.$$

Now, let $\frac{(5x-5)}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$

$$\Rightarrow \frac{(5x-5)}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow (5x-5) \equiv A(x-3) + B(x-2). \qquad \ldots \text{(i)}$$

Putting $x = 2$ on both sides of (i), we get $A = -5$.

Putting $x = 3$ on both sides of (i), we get $B = 10$.

$$\therefore \quad \frac{(x^2+1)}{(x^2-5x+6)} = 1 - \frac{5}{(x-2)} + \frac{10}{(x-3)}$$

$$\Rightarrow \int \frac{(x^2+1)}{(x^2-5x+6)}dx = \int dx - 5\int \frac{dx}{(x-2)} + 10\int \frac{dx}{(x-3)}$$

$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

EXAMPLE 3 *Evaluate* $\int \frac{(3x-2)}{(x+1)^2(x+3)}dx.$ **[CBSE 2006C]**

SOLUTION Let $\frac{(3x-2)}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$

$$\Rightarrow (3x-2) \equiv A(x+1)(x+3) + B(x+3) + C(x+1)^2. \qquad \ldots \text{(i)}$$

Putting $x = -3$ on both sides of (i), we get $C = \frac{-11}{4}$.

Putting $x = -1$ on both sides of (i), we get $B = \frac{-5}{2}$.

Comparing the coefficients of x^2 on both sides of (i), we get

$$A + C = 0 \Rightarrow A = -C = \frac{11}{4}.$$

$$\therefore \quad \frac{(3x-2)}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

$$\Rightarrow \int \frac{(3x-2)}{(x+1)^2(x+3)}dx = \frac{11}{4}\cdot\int\frac{dx}{(x+1)} - \frac{5}{2}\cdot\int\frac{1}{(x+1)^2}dx - \frac{11}{4}\cdot\int\frac{dx}{(x+3)}$$

$$= \frac{11}{4}\cdot\log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4}\cdot\log|x+3| + C$$

$$= \frac{11}{4}\cdot\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C.$$

EXAMPLE 4 *Evaluate* $\int\frac{dx}{(x^3+x^2+x+1)}$. **[CBSE 2006]**

SOLUTION We have $\frac{1}{(x^3+x^2+x+1)} = \frac{1}{x^2(x+1)+(x+1)} = \frac{1}{(x+1)(x^2+1)}$.

Let $\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$

$\Rightarrow 1 \equiv A(x^2+1) + (Bx+C)(x+1).$ … (i)

Putting $x = -1$ on both sides of (i), we get $A = \frac{1}{2}$.

Comparing the coefficients of x^2 on both sides of (i), we get

$$A + B = 0 \Rightarrow B = -A = \frac{-1}{2}.$$

Comparing the coefficients of x on both sides of (i), we get

$$B + C = 0 \Rightarrow C = -B = \frac{1}{2}.$$

$$\therefore \frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\therefore \int\frac{dx}{(x^3+x^2+x+1)} = \int\frac{dx}{(x+1)(x^2+1)}$$

$$= \frac{1}{2}\cdot\int\frac{dx}{(x+1)} - \frac{1}{2}\int\frac{x}{(x^2+1)}dx + \frac{1}{2}\int\frac{dx}{(x^2+1)}$$

$$= \frac{1}{2}\cdot\int\frac{dx}{(x+1)} - \frac{1}{4}\cdot\int\frac{2x}{(x^2+1)}dx + \frac{1}{2}\int\frac{dx}{(x^2+1)}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C.$$

EXAMPLE 5 *Evaluate* $\int\frac{x^4}{(x-1)(x^2+1)}dx.$

SOLUTION $\frac{x^4}{(x-1)(x^2+1)} = \frac{x^4}{(x^3-x^2+x-1)} = (x+1) + \frac{1}{(x^3-x^2+x-1)}$

$$\Rightarrow \quad \frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{(x-1)(x^2+1)}. \qquad \ldots \text{(i)}$$

Let $\dfrac{1}{(x-1)(x^2+1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+1)}$. Then,

$$1 \equiv A(x^2+1) + (Bx+C)(x-1). \qquad \ldots \text{(ii)}$$

Putting $x = 1$ in (ii), we get $A = \dfrac{1}{2}$.

Comparing the coefficients of x^2 on both sides of (ii), we get

$$A + B = 0 \;\Rightarrow\; B = -A = -\frac{1}{2}.$$

Comparing the constant terms on both sides of (ii), we get

$$A - C = 1 \;\Rightarrow\; C = (A-1) = \left(\frac{1}{2} - 1\right) = \frac{-1}{2}.$$

$$\therefore \quad \frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(x^2+1)}$$

$$\therefore \quad \frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\cdot\frac{(x+1)}{(x^2+1)}$$

$$\Rightarrow \int \frac{x^4}{(x-1)(x^2+1)}\,dx = \int (x+1)\,dx + \frac{1}{2}\int \frac{dx}{(x-1)}$$

$$-\frac{1}{4}\cdot\int \frac{2x}{(x^2+1)}\,dx - \frac{1}{2}\int \frac{dx}{(x^2+1)}$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C.$$

EXAMPLE 6 *Evaluate* $\displaystyle\int \frac{(3x+5)}{(x^3 - x^2 - x + 1)}\,dx.$ **[CBSE 2005C, '13C]**

SOLUTION $(x^3 - x^2 - x + 1) = x^2(x-1) - (x-1) = (x-1)(x^2-1) = (x-1)^2(x+1).$

Let $\dfrac{3x+5}{(x^3 - x^2 - x + 1)} = \dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+1)}$

$$\Rightarrow (3x+5) \equiv A(x-1)(x+1) + B(x+1) + C(x-1)^2. \qquad \ldots \text{(i)}$$

Putting $x = 1$ on both sides of (i), we get $B = 4$.

Putting $x = -1$ on both sides of (i), we get $C = \dfrac{1}{2}$.

Comparing the coefficient of x^2 on both sides of (i), we get

$$A + C = 0 \;\Rightarrow\; A = -C = \frac{-1}{2}.$$

$$\therefore \quad \frac{(3x+5)}{(x^3 - x^2 - x + 1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{(3x+5)}{(x^3-x^2-x+1)}dx = -\frac{1}{2}\int\frac{dx}{(x-1)} + 4\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{dx}{(x+1)}$$

$$= -\frac{1}{2}\log|x-1| - \frac{4}{(x-1)} + \frac{1}{2}\log|x+1| + C.$$

EXAMPLE 7 *Evaluate* $\int\frac{(x^3-1)}{(x^3+x)}dx.$

SOLUTION We have

$$\frac{(x^3-1)}{(x^3+x)} = 1 - \frac{(x+1)}{(x^3+x)} \qquad \text{[on dividing]}$$

$$= 1 - \frac{(x+1)}{x(x^2+1)}. \qquad \ldots \text{(i)}$$

Let $\frac{(x+1)}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}.$

Then, $(x+1) \equiv A(x^2+1) + (Bx+C)x.$ … (ii)

Putting $x = 0$ in (ii), we get $A = 1.$

Comparing the coefficients of x in (ii), we get $C = 1.$

Comparing the coefficients of x^2 in (ii), we get

$A + B = 0 \Rightarrow B = -A = -1.$

$\therefore \quad A = 1, \ B = -1$ and $C = 1.$

Thus, $\frac{(x+1)}{x(x^2+1)} = \frac{1}{x} + \frac{(1-x)}{(x^2+1)}$

$$\Rightarrow \int \frac{(x^3-1)}{(x^3+x)}dx = \int dx - \int\frac{(x+1)}{x(x^2+1)}dx$$

$$= x - \left\{\int\frac{dx}{x} + \int\frac{(1-x)}{(x^2+1)}dx\right\}$$

$$= x - \int\frac{dx}{x} - \int\frac{dx}{(x^2+1)} + \frac{1}{2}\int\frac{2x}{(x^2+1)}dx$$

$$= x - \log|x| - \tan^{-1}x + \frac{1}{2}\log(x^2+1) + C.$$

EXAMPLE 8 *Evaluate* $\int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx.$ **[CBSE 2006C]**

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\frac{dt}{(1-t)(2-t)}.$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$\Rightarrow 1 \equiv A(2-t) + B(1-t).$ … (i)

Putting $t = 1$ in (i), we get $A = 1.$

Putting $t = 2$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

$$= \int \left\{ \frac{1}{(1-t)} - \frac{1}{(2-t)} \right\} dt$$

$$= \int \frac{dt}{(1-t)} - \int \frac{dt}{(2-t)}$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C = \log\left|\frac{2-\sin x}{1-\sin x}\right| + C.$$

EXAMPLE 9 *Evaluate* $\int \frac{dx}{x\{6(\log x)^2 + 7\log x + 2\}}$.

SOLUTION Putting $\log x = t$ and $\frac{1}{x}dx = dt$, we get

$$I = \int \frac{dx}{x\{6(\log x)^2 + 7\log x + 2\}} = \int \frac{dt}{(6t^2 + 7t + 2)} = \int \frac{dt}{(2t+1)(3t+2)}.$$

Let $\frac{1}{(2t+1)(3t+2)} = \frac{A}{(2t+1)} + \frac{B}{(3t+2)}$.

Then, $1 \equiv A(3t+2) + B(2t+1)$. … (i)

Putting $t = -\frac{1}{2}$ in (i), we get $A = 2$.

Putting $t = \frac{-2}{3}$ in (i), we get $B = -3$.

$$\therefore \quad \frac{1}{(2t+1)(3t+2)} = \frac{2}{(2t+1)} - \frac{3}{(3t+2)}$$

$$\Rightarrow I = \int \frac{dt}{(2t+1)(3t+2)}$$

$$= \int \frac{2dt}{(2t+1)} - \int \frac{3dt}{(3t+2)}$$

$$= \log|2t+1| - \log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C$$

$$= \log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C.$$

EXAMPLE 10 *Evaluate* $\int \frac{x^2}{(1+x^3)(2+x^3)}dx.$ **[CBSE 2003C]**

SOLUTION Putting $x^3 = t$ and $x^2dx = \frac{1}{3}dt$, we get

$$I = \int \frac{x^2}{(1+x^3)(2+x^3)}dx = \frac{1}{3}\cdot\int \frac{dt}{(1+t)(2+t)}.$$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$. Then,

$$1 \equiv A(2+t) + B(1+t). \quad \ldots \text{(i)}$$

Putting $t = -1$ in (i), we get $A = 1$.

Putting $t = -2$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{(1+t)(2+t)} = \frac{1}{(1+t)} - \frac{1}{(2+t)}$$

$$\Rightarrow \quad I = \int \frac{dt}{(1+t)(2+t)} = \int \frac{dt}{(1+t)} - \int \frac{dt}{(2+t)}$$

$$= \log|1+t| - \log|2+t| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

$$= \log\left|\frac{1+x^3}{2+x^3}\right| + C.$$

EXAMPLE 11 *Evaluate* $\int \frac{dx}{(e^x - 1)}.$ **[CBSE 2003]**

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, i.e., $dx = \frac{1}{t}dt$, we get

$$I = \int \frac{dx}{(e^x - 1)} = \int \frac{dt}{t(t-1)}.$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$.

Then, $1 \equiv A(t-1) + Bt.$ … (i)

Putting $t = 0$ in (i), we get $A = -1$.

Putting $t = 1$ in (i), we get $B = 1$.

$$\therefore \quad \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{(t-1)}.$$

Hence, $I = \int \frac{dx}{(e^x - 1)}$

$$= \int \frac{dt}{t(t-1)} = \int \frac{-1}{t}dt + \int \frac{1}{(t-1)}dt$$

$$= -\log|t| + \log|t-1| + C$$

$$= \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{x(x^n+1)}$.

SOLUTION Putting $x^n = t$, we get $nx^{n-1}dx = dt$.

$$\therefore \quad \frac{nx^n}{x}dx = dt \;\Rightarrow\; \frac{1}{x}dx = \frac{1}{nt}dt \quad \text{(note)}.$$

$$\therefore \quad \int \frac{dx}{x(x^n+1)} = \int \frac{dt}{nt(t+1)} = \frac{1}{n}\cdot\int \frac{dt}{t(t+1)}. \qquad \ldots \text{(i)}$$

Let $\quad \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$.

Then, $1 \equiv A(t+1) + Bt.$ … (ii)

Putting $t = 0$ in (i), we get $A = 1$.

Putting $t = -1$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{t(t+1)} = \left\{\frac{1}{t} - \frac{1}{(t+1)}\right\}$$

$$\Rightarrow \int \frac{dx}{x(x^n+1)} = \frac{1}{n}\int \frac{dt}{t(t+1)}$$

$$= \frac{1}{n}\left[\int \frac{1}{t}dt - \int \frac{1}{(t+1)}dt\right]$$

$$= \frac{1}{n}\cdot\{\log|t| - \log|t+1|\} + C$$

$$= \frac{1}{n}\cdot\log\left|\frac{t}{t+1}\right| + C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{dx}{x(x^5+3)}$. **[CBSE 2013]**

SOLUTION Let $I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx.$ … (i)

Putting $x^5 = t$ and $5x^4dx = dt$ in (i), we get

$$I = \frac{1}{5}\int \frac{dt}{t(t+3)}. \qquad \ldots \text{(ii)}$$

Let $\frac{1}{t(t+3)} = \frac{A}{t} + \frac{B}{t+3}$.

Then, $1 \equiv A(t+3) + Bt$. … (iii)

Putting $t = 0$ on both sides of (iii), we get $A = \frac{1}{3}$.

Putting $t = -3$ on both sides of (iii), we get $B = \frac{-1}{3}$.

$$\therefore \quad \frac{1}{t(t+3)} = \frac{1}{3t} - \frac{1}{3(t+3)}$$

$$\Rightarrow \quad I = \frac{1}{5}\int\left\{\frac{1}{3t} - \frac{1}{3(t+3)}\right\}dt$$

$$= \frac{1}{15}\int\frac{1}{t}dt - \frac{1}{15}\int\frac{1}{(t+3)}dt$$

$$= \frac{1}{15}\log|t| - \frac{1}{15}\log|t+3| + C$$

$$= \frac{1}{15}\log|x^5| - \frac{1}{15}\log|x^5+3| + C.$$

EXAMPLE 14 *Evaluate* $\int\frac{x^2}{(x^2+4)(x^2+9)}dx$. **[CBSE 2013]**

SOLUTION Let $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$, where $x^2 = y$.

Let $\frac{y}{(y+4)(y+9)} = \frac{A}{(y+4)} + \frac{B}{(y+9)}$.

Then, $y \equiv A(y+9) + B(y+4)$. … (i)

Putting $y = -4$ on both sides of (i), we get $A = \frac{-4}{5}$.

Putting $y = -9$ on both sides of (i), we get $B = \frac{9}{5}$.

$$\therefore \quad \frac{y}{(y+4)(y+9)} = \frac{-4}{5(y+4)} + \frac{9}{5(y+9)}$$

$$\Rightarrow \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{-4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$\Rightarrow \quad \int\frac{x^2}{(x^2+4)(x^2+9)}dx = \frac{-4}{5}\int\frac{dx}{(x^2+4)} + \frac{9}{5}\int\frac{dx}{(x^2+9)}$$

$$= \left(\frac{-4}{5}\times\frac{1}{2}\right)\tan^{-1}\frac{x}{2} + \left(\frac{9}{5}\times\frac{1}{3}\right)\tan^{-1}\frac{x}{3} + C$$

$$= \frac{-2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C.$$

EXAMPLE 15 *Evaluate* $\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$.

SOLUTION Let $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{(x-4)} + \frac{B}{(x-5)} + \frac{C}{(x-6)}$.

Then, $(x-1)(x-2)(x-3)$

$\equiv (x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6)$
$+ C(x-4)(x-5).$ … (i)

Putting $x = 4$ on both sides of (i), we get $A = 3$.

Putting $x = 5$ on both sides of (i), we get $B = -24$.

Putting $x = 6$ on both sides of (i), we get $C = 30$.

$$\therefore \quad I = \int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx$$

$$= \int \left\{1 + \frac{3}{(x-4)} - \frac{24}{(x-5)} + \frac{30}{(x-6)}\right\} dx$$

$$= \int dx + 3\int \frac{dx}{(x-4)} - 24\int \frac{dx}{(x-5)} + 30\int \frac{dx}{(x-6)}$$

$$= x + 3\log|x-4| - 24\log|x-5| + 30\log|x-6| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx.$

SOLUTION We have

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)}, \text{ where } x^2 = t$$

$$= \frac{(t^2+3t+2)}{(t^2+7t+12)} = 1 - \frac{(4t+10)}{(t+3)(t+4)}.$$

Let $\frac{(4t+10)}{(t+3)(t+4)} = \frac{A}{(t+3)} + \frac{B}{(t+4)}$

$\Rightarrow (4t+10) \equiv A(t+4) + B(t+3).$ … (i)

Putting $t = -3$ in (i), we get $A = -2$.

Putting $t = -4$ in (i), we get $B = 6$.

$$\therefore \quad \frac{(4t+10)}{(t+3)(t+4)} = \frac{-2}{(t+3)} + \frac{6}{(t+4)}.$$ … (ii)

Thus, $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)}$, where $x^2 = t$

$$= \frac{(t^2+3t+2)}{(t^2+7t+12)} = 1 - \frac{(4t+10)}{(t+3)(t+4)}$$

$$= 1 - \left\{\frac{-2}{(t+3)} + \frac{6}{(t+4)}\right\} \quad \text{[from (ii)]}$$

$$= \left\{1 + \frac{2}{(t+3)} - \frac{6}{(t+4)}\right\}$$

$$= \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\}.$$

$$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

$$= \int dx + 2\int \frac{dx}{(x^2+3)} - 6\int \frac{dx}{(x^2+4)}$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{(3\sin\theta - 2)\cos\theta}{(5-\cos^2\theta - 4\sin\theta)}d\theta.$

SOLUTION We have

$$I = \int \frac{(3\sin\theta - 2)\cos\theta}{\{5-(1-\sin^2\theta) - 4\sin\theta\}}d\theta$$

$$= \int \frac{(3\sin\theta - 2)\cos\theta}{(4+\sin^2\theta - 4\sin\theta)}d\theta$$

$$= \int \frac{(3\sin\theta - 2)\cos\theta}{(\sin\theta - 2)^2}d\theta = \int \frac{(3t-2)}{(t-2)^2}dt, \text{ where } \sin\theta = t.$$

Let $\frac{(3t-2)}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}$. Then,

$$(3t-2) \equiv A(t-2) + B. \qquad \ldots \text{(i)}$$

Putting $t = 2$ in (i), we get $B = 4$.

Comparing the coefficients of t on both sides of (i), we get $A = 3$.

Thus, $A = 3$ and $B = 4$.

$$\therefore \frac{(3t-2)}{(t-2)^2} = \frac{3}{(t-2)} + \frac{4}{(t-2)^2}$$

$$\Rightarrow I = \int \frac{(3t-2)}{(t-2)^2}dt = \int \frac{3}{(t-2)}dt + \int \frac{4}{(t-2)^2}dt$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + C$$

$$= 3\log|\sin\theta - 2| - \frac{4}{(\sin\theta - 2)} + C$$

$$= 3\log(2-\sin\theta) + \frac{4}{(2-\sin\theta)} + C \quad [\because (2-\sin\theta) > 0].$$

EXAMPLE 18 *Evaluate* $\int \frac{(\tan\theta + \tan^3\theta)}{(1+\tan^3\theta)}d\theta.$ **[CBSE 2009C]**

SOLUTION We have

$$\frac{(\tan\theta + \tan^3\theta)}{(1+\tan^3\theta)} = \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan^3\theta)} = \frac{\tan\theta\sec^2\theta}{(1+\tan^3\theta)}.$$

$$\therefore \quad I = \int \frac{(\tan 0 + \tan^3 0)}{(1 + \tan^3 \theta)} d\theta$$

$$= \int \frac{\tan 0 \sec^2 0}{(1 + \tan^3 \theta)} d\theta$$

$$= \int \frac{t}{(1+t^3)} dt = \int \frac{t}{(1+t)(1-t+t^2)} dt, \text{ where } \tan \theta = t.$$

Let $\frac{t}{(1+t)(1-t+t^2)} = \frac{A}{(1+t)} + \frac{(Bt+C)}{(1-t+t^2)}$. Then,

$$t \equiv A(1-t+t^2) + (Bt+C)(1+t). \qquad \ldots \text{(i)}$$

Putting $t = -1$ on both sides of (i), we get $A = \frac{-1}{3}$.

Comparing the coefficients of t^2 on both sides of (i), we get

$$A + B = 0 \Rightarrow B = -A = \frac{1}{3}.$$

Comparing the constant terms on both sides of (i), we get

$$A + C = 0 \Rightarrow C = -A = \frac{1}{3}.$$

$$\therefore \quad \frac{t}{(1+t)(1-t+t^2)} = \frac{-1}{3(1+t)} + \frac{\left(\frac{1}{3}t + \frac{1}{3}\right)}{(1-t+t^2)}.$$

Now, $I = \int \frac{t}{(1+t)(1-t+t^2)} dt$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{2t}{(t^2-t+1)} dt + \frac{1}{3}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{(2t-1)+1}{(t^2-t+1)} dt + \frac{1}{3}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{(2t-1)}{(t^2-t+1)} dt + \frac{1}{2}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{dt}{\left(t^2-t+\frac{1}{4}\right)+\frac{3}{4}}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{dt}{(t-1/2)^2 + (\sqrt{3}/2)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{2}\right)}{(\sqrt{3}/2)} + C$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2 - t + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C.$$

EXAMPLE 19 *Evaluate* $\int \frac{dx}{(\sin x - \sin 2x)}$. **[CBSE 2010]**

SOLUTION $$\int \frac{dx}{(\sin x - \sin 2x)} = \int \frac{dx}{(\sin x - 2\sin x\cos x)}$$

$$= \int \frac{dx}{\sin x(1-2\cos x)} = \int \frac{\sin x}{\sin^2 x(1-2\cos x)}dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)}dx$$

$$= -\int \frac{dt}{(1-t^2)(1-2t)}, \text{ where } \cos x = t$$

$$= \int \frac{dt}{(t-1)(t+1)(1-2t)}. \quad \ldots \text{(i)}$$

Let $$\frac{1}{(t-1)(t+1)(1-2t)} = \frac{A}{(t-1)} + \frac{B}{(t+1)} + \frac{C}{(1-2t)}.$$

Then, $$1 \equiv A(t+1)(1-2t) + B(t-1)(1-2t) + C(t-1)(t+1). \quad \ldots \text{(ii)}$$

Putting $t = 1$ in (ii), we get $A = \frac{-1}{2}$.

Putting $t = -1$ in (ii), we get $B = \frac{-1}{6}$.

Putting $t = \frac{1}{2}$ in (ii), we get $C = \frac{-4}{3}$.

$$\therefore \quad I = -\frac{1}{2}\int \frac{dt}{(t-1)} - \frac{1}{6}\cdot\int \frac{dt}{(t+1)} - \frac{4}{3}\cdot\int \frac{dt}{(1-2t)}$$

$$= -\frac{1}{2}\log|t-1| - \frac{1}{6}\log|t+1| + \frac{2}{3}\cdot\int \frac{-2dt}{(1-2t)}$$

$$= -\frac{1}{2}\log|t-1| - \frac{1}{6}\log|t+1| + \frac{2}{3}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1-2\cos x| + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(1-\cos x)}{\cos x(1+\cos x)}dx.$

SOLUTION Let $\frac{(1-\cos x)}{\cos x(1+\cos x)} = \frac{(1-t)}{t(1+t)} = \frac{A}{t} + \frac{B}{(1+t)}$, where $t = \cos x$.

Then, $(1-t) \equiv A(1+t) + Bt.$

Putting $t = 0$ in this identity, we get $A = 1.$

Putting $t = -1$ in the identity, we get $B = -2.$

$$\therefore \quad \frac{(1-t)}{t(1+t)} = \frac{1}{t} - \frac{2}{1+t}$$

$$\text{or} \quad \frac{(1-\cos x)}{\cos x(1+\cos x)} = \frac{1}{\cos x} - \frac{2}{(1+\cos x)}$$

$$\therefore \int \frac{(1-\cos x)}{\cos x(1+\cos x)}dx = \int \frac{dx}{\cos x} - 2\int \frac{dx}{(1+\cos x)}$$

$$= \int \sec x\, dx - \int \sec^2 \frac{x}{2} dx$$

$$= \log|\sec x + \tan x| - 2\tan\frac{x}{2} + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(x^2+1)}{(x+1)^2}dx.$ **[CBSE 2006]**

SOLUTION On dividing (x^2+1) by (x^2+2x+1), we get

$$\frac{(x^2+1)}{(x+1)^2} = \left\{1 - \frac{2x}{(x+1)^2}\right\}.$$

Let $\frac{2x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$

$\Rightarrow \quad 2x \equiv A(x+1) + B.$ … (i)

On equating the coefficients of x, we get $A = 2.$

On equating constant terms, we get $A + B = 0 \Rightarrow B = -A = -2.$

$$\therefore \quad \frac{2x}{(x+1)^2} = \frac{2}{(x+1)} - \frac{2}{(x+1)^2}$$

$$\therefore \quad I = \int\left\{1 - \frac{2x}{(x+1)^2}\right\}$$

$$= \int\left\{1 - \frac{2}{(x+1)} + \frac{2}{(x+1)^2}\right\}dx$$

$$= x - 2\log|x+1| - \frac{2}{(x+1)} + C.$$

EXERCISE 15A

Evaluate:

1. $\int \frac{dx}{x(x+2)}$

2. $\int \frac{(2x+1)}{(x+2)(x-3)}dx$ **[CBSE 2007]**

3. $\int \frac{x}{(x+2)(3-2x)}dx$ **[CBSE 2003]** 4. $\int \frac{dx}{x(x-2)(x-4)}$

5. $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$ **[CBSE 2005]**

6. $\int \frac{(2x-3)}{(x^2-1)(2x+3)}dx$ **[CBSE 2002]** 7. $\int \frac{(2x+5)}{(x^2-x-2)}dx$

8. $\int \frac{(x^2+5x+3)}{(x^2+3x+2)}dx$ 9. $\int \frac{(x^2+1)}{(x^2-1)}dx$

10. $\int \frac{x^3}{(x^2-4)}dx$ 11. $\int \frac{(3+4x-x^2)}{(x+2)(x-1)}dx$

12. $\int \frac{x^3}{(x-1)(x-2)}dx$ 13. $\int \frac{(x^3-x-2)}{(1-x^2)}dx$

14. $\frac{(2x+1)}{(4-3x-x^2)}dx$ **[CBSE 2004C]** 15. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ **[CBSE 2011]**

16. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ **[CBSE 2003C, '07]** 17. $\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)}dx$

18. $\int \frac{\sin x \cos x}{(\cos^2 x - \cos x - 2)}dx$ 19. $\int \frac{e^x}{(e^{2x}+5e^x+6)}dx$

20. $\int \frac{e^x}{(e^{3x}-3e^{2x}-e^x+3)}dx$ 21. $\int \frac{2\log x}{x[2(\log x)^2 - \log x - 3]}dx$

22. $\int \frac{\text{cosec}^2 x}{(1-\cot^2 x)}dx$ 23. $\int \frac{\sec^2 x}{(\tan^3 x + 4\tan x)}dx$

24. $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx$ **[CBSE 2004]**

25. $\frac{e^x}{e^x(e^x-1)}dx$ 26. $\int \frac{dx}{x(x^4-1)}$

27. $\int \frac{(1-x^2)}{x(1-2x)}dx$ **[CBSE 2010]** 28. $\int \frac{(x^2+x+1)}{(x+2)(x+1)^2}dx$ **[CBSE 2006C]**

29. $\int \frac{(2x+9)}{(x+2)(x-3)^2}dx$ 30. $\int \frac{(x^2+1)}{(x-1)^2(x+3)}dx$ **[CBSE 2012]**

31. $\int \frac{(x^2+1)}{(x+3)(x-1)^2}dx$ 32. $\int \frac{(x^2+x+1)}{(x+2)(x^2+1)}dx$ **[CBSE 2009C]**

33. $\int \frac{2x}{(2x+1)^2}dx$ 34. $\int \frac{3x+1}{(x+2)(x-2)^2}dx$ **[CBSE 2007C]**

35. $\int \frac{(5x+8)}{x^2(3x+8)}dx$ 36. $\int \frac{(5x^2-18x+17)}{(x-1)^2(2x-3)}dx$

37. $\int \frac{8}{(x+2)(x^2+4)}dx$

38. $\int \frac{(3x+5)}{(x^3-x^2+x-1)}dx$

39. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

40. $\int \frac{x^2}{(x^4-1)}dx$

41. $\int \frac{dx}{(x^3-1)}$

42. $\int \frac{dx}{(x^3+1)}$

43. $\int \frac{dx}{(x+1)^2(x^2+1)}$ **[CBSE 2002]**

44. $\int \frac{17}{(2x+1)(x^2+4)}dx$

45. $\int \frac{dx}{(x^2+2)(x^2+4)}$

46. $\int \frac{x^2}{(x^2+4)(x^2+25)}dx$ **[CBSE 2013]**

47. $\int \frac{dx}{(e^x-1)^2}$

48. $\int \frac{dx}{x(x^5+1)}$

49. $\int \frac{dx}{x(x^6+1)}$

50. $\int \frac{dx}{\sin x(3+2\cos x)}$

51. $\int \frac{dx}{\cos x(5-4\sin x)}$

52. $\int \frac{dx}{\sin x\cos^2 x}$

53. $\int \frac{\tan x}{(1-\sin x)}dx$

54. $\int \frac{dx}{(\sin x+\sin 2x)}$

55. $\int \frac{x^2}{(x^4-x^2-12)}dx$

56. $\int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)}dx$

57. $\int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)}dx$ **[CBSE 2007]**

58. $\int \frac{2}{(1-x)(1+x^2)}dx$ **[CBSE 2012]**

59. $\int \frac{2x^2+1}{x^2(x^2+4)}dx$ **[CBSE 2013]**

ANSWERS (EXERCISE 15A)

1. $\frac{1}{2}\log\left|\frac{x}{x+2}\right|+C$

2. $\frac{3}{5}\log|x+2|+\frac{7}{5}\log|x-3|+C$

3. $\frac{-2}{7}\log|x+2|-\frac{3}{14}\log|3-2x|+C$

4. $\frac{1}{8}\log|x|-\frac{1}{4}\log|x-2|+\frac{1}{8}\log|x-4|+C$

5. $-\frac{1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+C$

6. $-\frac{1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{12}{5}\log|2x+3|+C$

7. $3\log|x-2|-\log|x+1|+C$

8. $x+3\log|x+2|-\log|x+1|+C$

9. $x+\log|x-1|-\log|x+1|+C$

10. $\frac{x^2}{2}+2\log|x^2-4|+C$

11. $-x+3\log|x+2|+2\log|x-1|+C$

12. $\frac{x^2}{2}+3x-\log|x-1|+8\log|x-2|+C$

13. $-\frac{x^2}{2}+\log\left|\frac{1-x}{1+x}\right|+C$

14. $-\frac{3}{5}\log|1-x|-\frac{7}{5}\log|4+x|+C$

15. $\frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right)+C$

16. $\log\left|\frac{1+\sin x}{2+\sin x}\right|+C$

17. $\log\left|\frac{2+\tan x}{3+\tan x}\right|+C$

18. $-\frac{1}{3}\log|\cos x+1|-\frac{2}{3}\log|\cos x-2|+C$

19. $\log\left|\frac{2+e^x}{3+e^x}\right|+C$

20. $\frac{1}{8}\log|e^x-3|-\frac{1}{4}\log|e^x-1|+\frac{1}{8}\log|e^x+1|+C$

21. $\frac{2}{5}\log|1+\log x|+\frac{3}{5}\log|2\log x-3|+C$

22. $-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right|+C$

23. $\frac{1}{4}\log|\tan x|-\frac{1}{8}\log|\tan^2 x+4|+C$

24. $-2\log|1+\sin x|+4\log|2+\sin x|+C$

25. $\log\left|\frac{e^x-1}{e^x}\right|+C$

26. $\log|x|-\frac{1}{4}\log|x^4+1|+C$

27. $\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C$

28. $3\log|x+2|-2\log|x+1|-\frac{1}{(x+1)}+C$

29. $\frac{1}{5}\log|x+2|-\frac{1}{5}\log|x-3|-\frac{3}{(x-3)}+C$

30. $\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$

31. $\frac{5}{2}\log|x-3|-\frac{3}{2}\log|x-1|+\frac{1}{(x-1)}+C$

32. $\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

33. $\frac{1}{2}\log|2x+1|+\frac{1}{2(2x+1)}+C$

34. $\frac{-5}{16}\log|x+2|+\frac{5}{16}\log|x-2|-\frac{7}{4(x-2)}+C$

35. $\frac{1}{4}\log|x|-\frac{1}{x}-\frac{1}{4}\log|3x+8|+C$

36. $\frac{4}{(x-1)}+\frac{5}{2}\log|2x-3|+C$

37. $\log|x+2|-\frac{1}{2}\log|x^2+4|+\tan^{-1}\frac{x}{2}+C$

38. $4\log|x-1|-2\log|x^2+1|-\tan^{-1}x+C$

39. $\frac{1}{2}\log|x^2+1|-\frac{1}{2}\log|x^2+3|+C$

40. $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}x+C$

41. $\frac{1}{3}\log|x-1|-\frac{1}{6}\log|x^2+x+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

42. $\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

43. $\frac{1}{2}\log|x+1|-\frac{1}{2(x+1)}-\frac{1}{4}\log|x^2+1|+C$

44. $2\log|2x+1|-\log|x^2+4|+2\tan^{-1}\frac{x}{2}+C$

45. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{1}{4}\tan^{-1}\frac{x}{2}+C$ **46.** $\frac{-1}{14}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)+C$

47. $x-\log|e^x-1|-\frac{1}{(e^x-1)}+C$ **48.** $\log|x|-\frac{1}{5}\log|x^5+1|+C$

49. $\log|x|-\frac{1}{6}\log|x^6+1|+C$

50. $-\frac{1}{2}\log|\cos x+1|+\frac{1}{10}\log|\cos x-1|+\frac{2}{5}\log|2\cos x+3|+C$

51. $-\frac{1}{2}\log|1-\sin x|+\frac{1}{18}\log|1+\sin x|+\frac{4}{9}\log|5-4\sin x|+C$

52. $\sec x-\frac{1}{2}\log\left|\frac{\cos x+1}{\cos x-1}\right|+C$ **53.** $\frac{1}{4}\log\left|\frac{1-\sin x}{1+\sin x}\right|+\frac{1}{2(1-\sin x)}+C$

54. $\frac{1}{2}\log|\cos x+1|+\frac{1}{6}\log|\cos x-1|-\frac{2}{3}\log|2\cos x+1|+C$

55. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C$

56. $\frac{1}{120}\tan^{-1}x-\frac{27}{56}\tan^{-1}\frac{x}{3}+\frac{64}{105}\tan^{-1}\frac{x}{4}+C$

57. $\frac{1}{2}\log|1-\cos 2x|-\frac{1}{2}\log|2-\cos 2x|+C$

58. $-\log(1-x)+\frac{1}{2}\log(1+x^2)+\tan^{-1}x+C$ **59.** $\frac{-1}{4x}+\frac{7}{8}\tan^{-1}\frac{x}{2}+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 15A)

7. $I=\int\frac{(2x+5)}{(x-2)(x+1)}dx.$ 8. $I=\int\left\{1+\frac{(2x+1)}{(x+2)(x+1)}\right\}dx.$

9. $I=\int\left\{1+\frac{2}{(x^2-1)}\right\}dx=\int\left\{1+\frac{2}{(x-1)(x+1)}\right\}dx.$

10. $I=\int\left\{x+\frac{4x}{(x^2-4)}\right\}dx=\int\left\{x+\frac{4x}{(x-2)(x+2)}\right\}dx.$

11. $\frac{(3+4x-x^2)}{(x+2)(x-1)}=\frac{(-x^2+4x+3)}{(x^2+x-2)}=\left\{-1+\frac{(5x+1)}{(x+2)(x-1)}\right\}.$

12. $\dfrac{x^3}{(x-1)(x-2)} = \dfrac{x^3}{(x^2-3x+2)} = \left[x + 3 + \dfrac{7x-6}{x^2-3x+2}\right] = \left\{x + 3 + \dfrac{7x-6}{(x-2)(x-1)}\right\}$.

13. $\dfrac{(x^3-x-2)}{(1-x^2)} = \dfrac{(x^3-x-2)}{(-x^2+1)} = \left\{-x - \dfrac{2}{(1-x^2)}\right\} = \left\{-x - \dfrac{2}{(1-x)(1+x)}\right\}$.

15. Putting $x^2 = t$ and $2x\,dx = dt$, we get
$$I = \int \frac{dt}{(2+t)(3+t)}.$$

16. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get
$$I = \int \frac{dt}{(1+t)(2+t)}.$$

17. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

18. Put $\cos x = t$ and $-\sin x\,dx = dt$.

19. Put $e^x = t$ and $e^x dx = dt$.

20. Put $e^x = t$ and $e^x dx = dt$.

21. Put $\log x = t$ and $\dfrac{1}{x}dx = dt$.

22. Put $\cot x = t$ and $-\text{cosec}^2 x\,dx = dt$.

23. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

24. Write $\sin 2x = 2\sin x \cos x$. Put $\sin x = t$ and $\cos x\,dx = dt$.

25. Put $e^x = t$ and $e^x dx = dt$.

26. $I = \dfrac{x^3}{x^4(x^4+1)}\,dx$. Put $x^4 = t$ and $4x^3 dx = dt$.

27. $\dfrac{(1-x^2)}{x(1-2x)} = \dfrac{(-x^2+1)}{(-2x^2+x)} = \left\{\dfrac{1}{2} + \dfrac{\left(1-\dfrac{1}{2x}\right)}{x(1-2x)}\right\}$.

Let $\dfrac{\left(1-\dfrac{1}{2x}\right)}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{(1-2x)}$. Then, $\left(1-\dfrac{1}{2x}\right) \equiv A(1-2x) + Bx$.

$(x = 0 \Rightarrow A = 1)$ and $\left(x = \dfrac{1}{2} \Rightarrow B = \dfrac{3}{2}\right)$

$\therefore\ I = \int \left\{\dfrac{1}{2} + \dfrac{1}{x} + \dfrac{3}{2(1-2x)}\right\} dx.$

32. Let $\dfrac{(x^2+x+1)}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x+2)}$.

33. $\dfrac{2x}{(2x+1)^2} = \dfrac{A}{(2x+1)} + \dfrac{B}{(2x+1)^2}$.

34. Let $\dfrac{(3x+5)}{(x-2)^2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2}$.

35. Let $\frac{(5x+8)}{x^2(3x+8)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(3x+8)}$.

39. Put $x^2 = t$ and $2x\,dx = dt$. Then,

$$I = \int \frac{dt}{(t+1)(t+3)}. \quad \ldots \text{(i)}$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$. Then,

$1 \equiv A(t+3) + B(t+1)$. … (ii)

Putting $t = -1$ in (ii), we get $A = \frac{1}{2}$.

Putting $t = -3$ in (ii), we get $B = \frac{-1}{2}$.

43. Let $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$.

$\therefore \quad 1 \equiv A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$. … (i)

Putting $x = -1$ in (i), we get $B = \frac{1}{2}$.

Comparing coefficients of various powers of x, we get

$A + B + D = 0,\ A + C + 2D = 0,\ A + B + 2C + D = 0,\ A + C = 0$

$(A + C = 0,\ A + C + 2D = 0) \Rightarrow D = 0.$

$\left[A + B + D = 0 \Rightarrow A + B = 0 \Rightarrow A = \frac{-1}{2}\right]$ and $\left(C = -A = \frac{1}{2}\right)$.

45. Put $x^2 = t$. Then,

$$\frac{1}{(x^2+2)(x^2+4)} = \frac{1}{(t+2)(t+4)} = \frac{A}{(t+2)} + \frac{B}{(t+4)}.$$

$\therefore \quad 1 \equiv A(t+4) + B(t+2)$. … (i)

Putting $t = -2$ in (i), we get $A = \frac{1}{2}$.

Putting $t = -4$ in (i), we get $B = \frac{-1}{2}$.

$\therefore \quad I = \frac{1}{2}\int \frac{dx}{(x^2+2)} - \frac{1}{2}\int \frac{dx}{(x^2+4)}$.

46. Let $\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{y+1}{(y+4)(y+25)}$, where $x^2 = y$.

Let $\frac{y+1}{(y+4)(y+25)} = \frac{A}{(y+4)} + \frac{B}{(y+25)}$.

Then, $y + 1 \equiv A(y+25) + B(y+4)$. … (i)

Putting $y = -4$ in (i), we get $A = \frac{-3}{21} = \frac{-1}{7}$.

Putting $y = -25$ in (i), we get $B = \frac{-24}{-21} = \frac{8}{7}$.

$\therefore \quad I = \frac{-1}{7}\int \frac{dx}{(x^2+4)} + \frac{8}{7}\int \frac{dx}{(x^2+25)}$.

$$= \left(\frac{-1}{7} \times \frac{1}{2}\right) \tan^{-1} \frac{x}{2} + \left(\frac{8}{7} \times \frac{1}{5}\right) \tan^{-1} \frac{x}{5} + C$$

$$= \frac{-1}{14} \tan^{-1} \frac{x}{2} + \frac{8}{35} \tan^{-1} \frac{x}{5} + C.$$

47. Put $e^x - 1 = t$ and $e^x dx = dt$, i.e., $dx = \frac{dt}{(t+1)}$.

$\therefore \quad I = \int \frac{dt}{t^2(t+1)}$. Let $\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$.

Then, $1 \equiv At(t+1) + B(t+1) + Ct^2$. … (i)

48. $I = \int \frac{x^4}{x^5(x^5+1)} dx$. Put $x^5 = t$ and $5x^4\, dx = dt$.

49. $I = \int \frac{x^5}{x^6(x^6+1)} dx$. Now, put $x^6 = t$ and $6x^5\, dx = dt$.

50. $I = \int \frac{\sin x}{\sin^2 x(3 + 2\cos x)} dx = \int \frac{\sin x}{(1 - \cos^2 x)(3 + 2\cos x)} dx.$

Now, put $\cos x = t$ and $-\sin x\, dx = dt$.

51. $I = \int \frac{\cos x}{\cos^2 x(5 - 4\sin x)} dx = \int \frac{\cos x}{(1 - \sin^2 x)(5 - 4\sin x)} dx.$

Now, put $\sin x = t$ and $\cos x\, dx = dt$.

52. $I = \int \frac{\sin x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin x}{(1 - \cos^2 x)\cos^2 x} dx.$

Put $\cos x = t$ and $-\sin x\, dx = dt$.

53. $I = \int \frac{\sin x}{\cos x(1 - \sin x)} dx = \int \frac{\sin x \cos x}{\cos^2 x(1 - \sin x)} dx$

$= \int \frac{\sin x \cos x}{(1 - \sin^2 x)(1 - \sin x)} dx = \frac{t}{(1 - t^2)(1 - t)} dt$, where $\sin x = t$

$= \int \frac{t}{(1+t)(1-t)^2} dt.$

54. $I = \int \frac{dx}{(\sin x + 2\sin x \cos x)} = \int \frac{dx}{\sin x(1 + 2\cos x)}$

$= \int \frac{\sin x}{(1 - \cos^2 x)(1 + 2\cos x)} dx$. Now put $\cos x = t$.

55. Let $x^2 = t$. Then, $\frac{x^2}{(x^4 - x^2 - 12)} = \frac{t}{(t^2 - t - 12)} = \frac{t}{(t-4)(t+3)}$.

Let $\frac{t}{(t-4)(t+3)} = \frac{A}{(t-4)} + \frac{B}{(t+3)}$.

Then, $t \equiv A(t+3) + B(t-4)$.

Putting $t = -3$, we get $B = \frac{3}{7}$.

Putting $t = 4$, we get $A = \frac{4}{7}$.

$\therefore \quad \dfrac{x^2}{(x^4 - x^2 - 12)} = \dfrac{4}{7} \cdot \dfrac{1}{(x^2 - 4)} + \dfrac{3}{7} \cdot \dfrac{1}{(x^2 + 3)}$.

Now, integrate.

56. Let $x^2 = t$. Then, $\dfrac{x^4}{(x^2 + 1)(x^2 + 9)(x^2 + 16)} = \dfrac{t^2}{(t + 1)(t + 9)(t + 16)}$.

EXERCISE 15B

Very-Short-Answer Questions

Evaluate:

1. $\int x^{-6} dx$
2. $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right) dx$
3. $\int \sin\ 3x\, dx$
4. $\int \dfrac{x^2}{(1 + x^3)} dx$ **[CBSE 2008]**
5. $\int \dfrac{2\cos x}{3\sin^2 x} dx$ **[CBSE 2008]**
6. $\int \dfrac{(3\sin \phi - 2)\cos \phi}{(5 - \cos^2 \phi - 4\sin \phi)} d\phi$ **[CBSE 2007]**
7. $\int \sin^2 x\, dx$
8. $\int \dfrac{(\log x)^2}{x} dx$
9. $\int \dfrac{(x+1)(x + \log x)^2}{x} dx$
10. $\int \dfrac{\sin x}{(1 + \cos x)} dx$
11. $\int \dfrac{(1 + \tan x)}{(1 - \tan x)} dx$
12. $\int \dfrac{(1 - \cot x)}{(1 + \cot x)} dx$
13. $\int \dfrac{(1 + \cot x)}{(x + \log \sin x)} dx$
14. $\int \dfrac{(1 - \sin 2x)}{(x + \cos^2 x)} dx$
15. $\int \dfrac{\sec^2 (\log x)}{x} dx$
16. $\int \dfrac{\sin (2\tan^{-1} x)}{(1 + x^2)} dx$
17. $\int \dfrac{\tan x \sec^2 x}{(1 - \tan^2 x)} dx$
18. $\int \dfrac{(x^4 + 1)}{(x^2 + 1)} dx$
19. $\int \tan^{-1} \sqrt{\dfrac{1 - \sin x}{1 + \sin x}}\, dx$
20. $\int \log (1 + x^2)\, dx$
21. $\int \cos x \cos\ 3x\, dx$
22. $\int \sin\ 3x \sin x\, dx$
23. $\int \dfrac{xe^x}{(x+1)^2} dx$
24. $\int e^x \{\tan x - \log \cos x\}\, dx$
25. $\int \dfrac{dx}{(1 - \sin x)}$
26. $\int x \cos x^2 dx$

27. $\int \frac{\cot x}{\sqrt{\sin x}}\,dx$

28. $\int \frac{\sec^2 x}{\text{cosec}^2 x}\,dx$

29. $\int \sin^{-1}(\cos x)\,dx$

30. $\int \frac{dx}{(\sqrt{x+2}+\sqrt{x+1})}$

31. $\int 2^x\,dx$

32. $\int \frac{(1+\tan x)}{(x+\log \sec x)}\,dx$

33. $\int \frac{\sec^2(\log x)}{x}\,dx$

34. $\int (2x+1)(\sqrt{x^2+x+1})\,dx$

35. $\int \frac{dx}{\sqrt{9x^2+16}}$

36. $\int \frac{dx}{\sqrt{4-9x^2}}$

37. $\int \frac{dx}{\sqrt{4x^2-25}}$

38. $\int \sqrt{4-x^2}\,dx$

39. $\int \sqrt{9+x^2}\,dx$

40. $\int \sqrt{x^2-16}\,dx$

ANSWERS (EXERCISE 15B)

1. $\frac{-1}{5x^5}+C$

2. $\frac{2}{3}x^{3/2}+2\sqrt{x}+C$

3. $\frac{-1}{3}\cos 3x+C$

4. $\frac{1}{3}\log|1+x^3|+C$

5. $\frac{-2}{3}\text{cosec}\,x+C$

6. $3\log|\sin\phi-2|-\frac{4}{(\sin\phi-2)}+C$

7. $\frac{1}{2}x-\frac{1}{4}\sin 2x+C$

8. $\frac{1}{3}(\log x)^3+C$

9. $\frac{1}{3}(x+\log x)^3+C$

10. $-\log|1+\cos x|+C$

11. $-\log|\cos x+\sin x|+C$

12. $-\log|\sin x+\cos x|+C$

13. $\log|x+\log\sin x|+C$

14. $\log|x+\cos^2 x|+C$

15. $\tan(\log x)+C$

16. $-\frac{1}{2}\cos(2\tan^{-1}x)+C$

17. $-\frac{1}{2}\log|1-\tan^2 x|+C$

18. $\frac{x^3}{3}-x+2\tan^{-1}x+C$

19. $\frac{\pi x}{4}-\frac{x^2}{4}+C$

20. $x\log(1+x^2)-2x+2\tan^{-1}x+C$

21. $\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$

22. $\frac{\sin 2x}{4}-\frac{\sin 4x}{8}+C$

23. $\frac{e^x}{(x+1)}+C$

24. $-e^x\log\cos x+C$

25. $\tan x+\sec x+C$

26. $\frac{1}{2}\sin x^2+C$

27. $\frac{-2}{\sqrt{\sin x}} + C$

28. $\tan x - x + C$

29. $\frac{\pi x}{2} - \frac{x^2}{2} + C$

30. $\frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + C$

31. $\frac{2^x}{\log 2} + C$

32. $\log|x + \log \sec x| + C$

33. $\tan(\log x) + C$

34. $\frac{2}{3}(x^2 + x + 1)^{3/2} + C$

35. $\frac{1}{3}\log\left|3x + \sqrt{9x^2+16}\right| + C$

36. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$

37. $\frac{1}{2}\log\left|2x + \sqrt{4x^2-25}\right| + C$

38. $\frac{x}{2}\cdot\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

39. $\frac{x}{2}\sqrt{9+x^2} + \frac{9}{2}\log\left|x + \sqrt{9+x^2}\right| + C$

40. $\frac{1}{2}x\sqrt{x^2-16} - 8\log\left|x + \sqrt{x^2-16}\right| + C$

OBJECTIVE QUESTIONS I

Mark (✓) against the correct answer in each of the following:

1. $\int \frac{dx}{(9+x^2)} = ?$

   (a) $\tan^{-1}\frac{x}{3} + C$ (b) $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ (c) $3\tan^{-1}\frac{x}{3} + C$ (d) none of these

2. $\int \frac{dx}{(4+16x^2)} = ?$

   (a) $\frac{1}{32}\tan^{-1}4x + C$ (b) $\frac{1}{16}\tan^{-1}\frac{x}{2} + C$

   (c) $\frac{1}{8}\tan^{-1}2x + C$ (d) $\frac{1}{4}\tan^{-1}\frac{x}{2} + C$

3. $\int \frac{dx}{(9+4x^2)}dx = ?$

   (a) $\frac{1}{2}\tan^{-1}\frac{2x}{3} + C$ (b) $\frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

   (c) $\frac{1}{6}\tan^{-1}\frac{3x}{2} + C$ (d) none of these

4. $\int \frac{\sin x}{(1+\cos^2 x)}dx = ?$

   (a) $-\tan^{-1}(\cos x) + C$ (b) $\cot^{-1}(\cos x) + C$

   (c) $-\cot^{-1}(\cos x) + C$ (d) $\tan^{-1}(\cos x) + C$

5. $\int \frac{\cos x}{(1+\sin^2 x)} dx = ?$

(a) $-\tan^{-1}(\sin\ x) + C$ (b) $\tan^{-1}(\cos x) + C$

(c) $\tan^{-1}(\sin\ x) + C$ (d) $-\tan^{-1}(\cos x) + C$

6. $\int \frac{e^x}{(e^{2x}+1)} dx = ?$

(a) $\cot^{-1}(e^x) + C$ (b) $\tan^{-1}(e^x) + C$ (c) $2\tan^{-1}(e^x) + C$ (d) none of these

7. $\int \frac{3x^5}{(1+x^{12})} dx = ?$

(a) $\tan^{-1} x^6 + C$ (b) $\frac{1}{4}\tan^{-1} x^6 + C$ (c) $\frac{1}{2}\tan^{-1} x^6 + C$ (d) none of these

8. $\int \frac{2x^3}{(4+x^8)} dx = ?$

(a) $\frac{1}{2}\tan^{-1}\frac{x^4}{2} + C$ (b) $\frac{1}{4}\tan^{-1}\frac{x^4}{2} + C$

(c) $\frac{1}{2}\tan^{-1} x^4 + C$ (d) none of these

9. $\int \frac{dx}{(x^2+4x+8)} = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

(c) $\frac{1}{2}\tan^{-1}(x+2) + C$ (d) $\tan^{-1}\left(\frac{x+2}{2}\right) + C$

10. $\int \frac{dx}{(2x^2+x+3)} = ?$

(a) $\frac{1}{\sqrt{23}}\tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right) + C$ (b) $\frac{1}{\sqrt{23}}\tan^{-1}\left(\frac{x+1}{\sqrt{23}}\right) + C$

(c) $\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right) + C$ (d) none of these

11. $\int \frac{dx}{(e^x + e^{-x})} = ?$

(a) $\tan^{-1}(e^x) + C$ (b) $\tan^{-1}(e^{-x}) + C$

(c) $-\tan^{-1}(e^{-x}) + C$ (d) none of these

12. $\int \frac{x^2}{(9+4x^2)} = ?$

(a) $\frac{x}{4} - \frac{1}{8}\tan^{-1}\frac{x}{3} + C$ (b) $\frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{x}{3} + C$

(c) $\frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{2x}{3} + C$ (d) none of these

13. $\int \frac{(x^2-1)}{(x^2+4)}dx = ?$

(a) $x - 5\tan^{-1}\frac{x}{2} + C$ (b) $x - \frac{5}{2}\tan^{-1}\frac{x}{2} + C$

(c) $x - \frac{5}{2}\tan^{-1}\frac{5x}{2} + C$ (d) none of these

14. $\int \frac{dx}{(4+9x^2)} = ?$

(a) $\frac{2}{3}\tan^{-1}\frac{3x}{2} + C$ (b) $\frac{1}{6}\tan^{-1}3x + C$

(c) $\frac{1}{6}\tan^{-1}\frac{3x}{2} + C$ (d) none of these

15. $\int \frac{dx}{(4x^2 - 4x + 3)} = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$ (b) $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

(c) $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$ (d) none of these

16. $\int \frac{dx}{(\sin^4 x + \cos^4 x)} = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}\tan x}\right) + C$ (d) none of these

17. $\int \frac{(x^2+1)}{(x^4 + x^2 + 1)}dx = ?$

(a) $\tan^{-1}\frac{(x^2-1)}{\sqrt{3}} + C$ (b) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}} + C$

(c) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}x} + C$ (d) none of these

18. $\int \frac{\sin 2x}{(\sin^4 x + \cos^4 x)}dx = ?$

(a) $\tan^{-1}(\tan^2 x) + C$ (b) $x^2 + C$

(c) $-\tan^{-1}(\tan^2 x) + C$ (d) none of these

19. $\int \frac{dx}{(1-9x^2)} = ?$

(a) $\frac{1}{3}\log\left|\frac{1+3x}{1-3x}\right| + C$ (b) $\frac{1}{3}\log\left|\frac{1-3x}{1+3x}\right| + C$

(c) $\frac{1}{6}\log\left|\frac{1+3x}{1-3x}\right| + C$ (d) $\frac{1}{6}\log\left|\frac{1-3x}{1+3x}\right| + C$

20. $\int \frac{dx}{(16-4x^2)} = ?$

(a) $\frac{1}{8}\log\left|\frac{2-x}{2+x}\right| + C$ (b) $\frac{1}{16}\log\left|\frac{2-x}{2+x}\right| + C$

(c) $\frac{1}{8}\log\left|\frac{2+x}{2-x}\right| + C$ (d) $\frac{1}{16}\log\left|\frac{2+x}{2-x}\right| + C$

21. $\int \frac{x^2}{(1-x^6)}dx = ?$

(a) $\frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$ (b) $\frac{1}{6}\log\left|\frac{1-x^3}{1+x^3}\right| + C$

(c) $\frac{1}{3}\log\left|\frac{1-x^3}{1+x^3}\right| + C$ (d) none of these

22. $\int \frac{x}{(1-x^4)}dx = ?$

(a) $\frac{1}{4}\log\left|\frac{1+x^2}{1-x^2}\right| + C$ (b) $\frac{1}{4}\log\left|\frac{1-x^2}{1+x^2}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{1+x^2}{1-x^2}\right| + C$ (d) none of these

23. $\int \frac{x^2}{(a^6-x^6)}dx = ?$

(a) $\frac{1}{3a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right| + C$ (b) $\frac{1}{6a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right| + C$

(c) $\frac{1}{6a^3}\log\left|\frac{a^3-x^3}{a^3+x^3}\right| + C$ (d) none of these

24. $\int \frac{dx}{(3-2x-x^2)} = ?$

(a) $\frac{1}{4}\log\left|\frac{3+x}{3-x}\right| + C$ (b) $\frac{1}{4}\log\left|\frac{1+x}{1-x}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{3+x}{1-x}\right| + C$ (d) none of these

25. $\int \frac{dx}{(\cos^2 x - 3\sin^2 x)} = ?$

(a) $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$ (b) $\frac{1}{\sqrt{3}}\log\left|\frac{1-\sqrt{3}\tan x}{1+\sqrt{3}\tan x}\right| + C$

(c) $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C$ (d) none of these

26. $\int \frac{\text{cosec}^2 x}{(1-\cot^2 x)} dx = ?$

(a) $\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C$ (b) $-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{1-\cot x}{1+\cot x}\right| + C$ (d) none of these

27. $\int \frac{dx}{(4x^2-1)} = ?$

(a) $\frac{1}{2}\log\left|\frac{2x-1}{2x+1}\right| + C$ (b) $\frac{1}{2}\log\left|\frac{2x+1}{2x-1}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C$ (d) none of these

28. $\int \frac{x}{(x^4-16)} dx = ?$

(a) $\frac{1}{4}\log\left|\frac{x^2+4}{x^2-4}\right| + C$ (b) $\frac{1}{16}\log\left|\frac{x^2+4}{x^2-4}\right| + C$

(c) $\frac{1}{16}\log\left|\frac{x^2-4}{x^2+4}\right| + C$ (d) none of these

29. $\int \frac{dx}{(\sin^2 x - 4\cos^2 x)} = ?$

(a) $\frac{1}{4}\log\left|\frac{\tan x - 2}{\tan x + 2}\right| + C$ (b) $\frac{1}{4}\log\left|\frac{\tan x + 2}{\tan x - 2}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{1-\tan x}{1+\tan x}\right| + C$ (d) none of these

30. $\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)} = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right) + C$ (b) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right) + C$

(c) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C$ (d) none of these

31. $\int \frac{\sin x}{\sin 3x} dx = ?$

(a) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x}{\sqrt{3}-\sin x}\right| + C$ (b) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\cos x}{\sqrt{3}-\cos x}\right| + C$

(c) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$ (d) none of these

32. $\int \frac{(x^2+1)}{(x^4+1)}dx = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+C$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$

(c) $\frac{1}{\sqrt{2}}\log\left(\frac{x^2+1}{x^2-1}\right)+C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS I)

1. (b) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (b) **7.** (c) **8** (b) **9.** (a) **10.** (c)
11. (a) **12.** (c) **13.** (b) **14.** (c) **15.** (b) **16.** (a) **17.** (c) **18.** (a) **19.** (c) **20.** (d)
21. (a) **22.** (a) **23.** (b) **24.** (c) **25.** (c) **26.** (b) **27.** (c) **28.** (c) **29.** (a) **30.** (c)
31. (c) **32.** (b)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS I

1. $I = \int \frac{dx}{(3^2 + x^2)} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$

2. $I = \frac{1}{16}\cdot\int \frac{dx}{\left(\frac{1}{4}+x^2\right)} = \frac{1}{16}\cdot\int \frac{dx}{\left\{\left(\frac{1}{2}\right)^2 + x^2\right\}} = \frac{1}{16}\cdot\frac{1}{(1/2)}\tan^{-1}\frac{x}{(1/2)} + C = \frac{1}{8}\tan^{-1}2x + C.$

3. $I = \frac{1}{4}\int \frac{dx}{\left(\frac{9}{4}+x^2\right)} = \frac{1}{4}\cdot\int \frac{dx}{\left\{\left(\frac{3}{2}\right)^2 + x^2\right\}}$

$= \frac{1}{4}\cdot\frac{1}{(3/2)}\tan^{-1}\frac{x}{(3/2)} + C = \frac{1}{6}\tan^{-1}\frac{2x}{3} + C.$

4. Putting $\cos x = t$ and $-\sin x\,dx = dt$, we get

$$I = -\int \frac{dt}{(1+t^2)} = -\tan^{-1}t + C = -\tan^{-1}(\cos x) + C.$$

5. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \frac{dt}{(1+t^2)} = \tan^{-1}t + C = \tan^{-1}(\sin x) + C.$$

6. Putting $e^x = t$ and $e^x dx = dt$, we get

$$I = \int \frac{dt}{(t^2+1)} = \tan^{-1}t + C = \tan^{-1}(e^x) + C.$$

7. Putting $x^6 = t$ and $6x^5 dx = dt$, i.e., $3x^5 dx = \frac{1}{2} dt$, we get

$$I = \frac{1}{2}\int \frac{dt}{(1+t^2)} = \frac{1}{2}\tan^{-1} t + C = \frac{1}{2}\tan^{-1} x^6 + C.$$

8. Putting $x^4 = t$ and $4x^3 dx = dt$, i.e., $2x^3 dx = \frac{1}{2} dt$, we get

$$I = \frac{1}{2}\int \frac{dt}{(2^2 + t^2)} = \frac{1}{2} \times \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{4}\tan^{-1}\frac{x^4}{2} + C.$$

9. $I = \int \frac{dx}{\{(x+2)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}$, where $x + 2 = t$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C.$$

10. $I = \frac{1}{2}\int \frac{dx}{\left(x^2 + \frac{x}{2} + \frac{3}{2}\right)} = \frac{1}{2}\int \frac{dx}{\left\{x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2\right\} + \left(\frac{3}{2} - \frac{1}{16}\right)}$

$$= \frac{1}{2}\int \frac{dx}{\left\{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2\right\}} = \frac{1}{2}\int \frac{dx}{\left\{t^2 + \left(\frac{\sqrt{23}}{4}\right)\right\}}, \text{ where } \left(x + \frac{1}{4}\right) = t$$

$$= \frac{1}{2} \times \frac{4}{\sqrt{23}}\tan^{-1}\frac{t}{\left(\frac{\sqrt{23}}{4}\right)} + C = \frac{2}{\sqrt{23}}\tan^{-1}\frac{4t}{\sqrt{23}} + C = \frac{2}{\sqrt{23}}\tan^{-1}\frac{4\left(x + \frac{1}{4}\right)}{\sqrt{23}} + C$$

$$= \frac{2}{\sqrt{23}}\tan^{-1}\frac{(4x+1)}{\sqrt{23}} + C.$$

11. $I = \int \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int \frac{e^x dx}{(e^{2x} + 1)} = \int \frac{dt}{(t^2 + 1)}$, where $e^x = t$

$$= \tan^{-1} t + C = \tan^{-1}(e^x) + C.$$

12. On dividing x^2 by $(4x^2 + 9)$, we get $\frac{1}{4} - \frac{(9/4)}{(4x^2 + 9)}$.

$$\therefore \quad I = \frac{1}{4}\int dx - \frac{9}{4} \cdot \frac{1}{4}\int \frac{dx}{\left(x^2 + \frac{9}{4}\right)} = \frac{x}{4} - \frac{9}{16} \cdot \frac{1}{(3/2)}\tan^{-1}\frac{x}{(3/2)} + C$$

$$= \frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{2x}{3} + C.$$

13. On dividing, we get $\frac{(x^2 - 1)}{(x^2 + 4)} = \left\{1 - \frac{5}{(x^2 + 4)}\right\}$.

$$\therefore \quad I = \int dx - 5\int \frac{dx}{(x^2 + 4)} = x - \frac{5}{2}\tan^{-1}\frac{x}{2} + C.$$

14. $I = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{9} + x^2\right)} = \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{1}{3}\right)^2 + x^2\right\}}$

$= \frac{1}{9} \cdot \frac{1}{(2/3)} \tan^{-1} \frac{x}{(2/3)} + C = \frac{1}{6} \tan^{-1} \frac{3x}{2} + C.$

15. $I = \frac{1}{4}\int \frac{dx}{\left(x^2 - x + \frac{3}{4}\right)} = \frac{1}{4}\int \frac{dx}{\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{2}} = \frac{1}{4}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

$= \frac{1}{4}\int \frac{dt}{\left\{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right\}} = \frac{1}{4} \cdot \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \tan^{-1} \frac{t}{\left(\frac{1}{\sqrt{2}}\right)} + C = \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2}\, t) + C$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left\{\frac{\sqrt{2}(2x-1)}{2}\right\} + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C.$

16. $\frac{1}{(\sin^4 x + \cos^4 x)} = \frac{\sec^4 x}{(\tan^4 x + 1)} = \frac{(1 + \tan^2 x)\sec^2 x}{(\tan^4 x + 1)}$

Put $\tan x = t$ and $\sec^2 x\, dx = dt$.

$\therefore \quad I = \int \frac{(1+t^2)}{(1+t^4)} dt = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2} dt = \int \frac{du}{u^2 + (\sqrt{2})^2}$, where $\left(t - \frac{1}{t}\right) = u$

$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{(t^2 - 1)}{\sqrt{t}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \frac{(\tan^2 x - 1)}{\sqrt{2} \tan x} + C.$

17. $\frac{(x^2+1)}{(x^4 + x^2 + 1)} = \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} = \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2}.$

Put $\left(x - \frac{1}{x}\right) = t$ and $\left(1 + \frac{1}{x^2}\right) dx = dt.$

$\therefore \quad I = \int \frac{dt}{t^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + C = \frac{1}{\sqrt{3}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} + C$

$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{(x^2 - 1)}{\sqrt{3}x} + C.$

18. On dividing Nr and Dr by $\cos^4 x$, we get

$$I = \int \frac{\frac{2 \sin x}{\cos^3 x}}{(\tan^4 x + 1)} dx = 2\int \frac{\tan x \sec^2 x}{(\tan^4 x + 1)} dx$$

$$= 2\int \frac{t}{(t^4+1)}dt, \text{ where } \tan x = t$$

$$= \frac{du}{(u^2+1)} = \tan^{-1}u + C, \text{ where } t^2 = u$$

$$= \tan^{-1}t^2 + C = \tan^{-1}(\tan^2 x) + C.$$

19. $I = \frac{1}{9}\cdot\int \frac{dx}{\left(\frac{1}{9}-x^2\right)} = \frac{1}{9}\cdot\int \frac{dx}{\left\{\left(\frac{1}{3}\right)^2 - x^2\right\}}$

$$= \frac{1}{9}\cdot\frac{1}{\left(2\times\frac{1}{3}\right)}\log\left|\frac{\frac{1}{3}+x}{\frac{1}{3}-x}\right| + C = \frac{1}{6}\log\left|\frac{1+3x}{1-3x}\right| + C.$$

20. $I = \frac{1}{4}\int \frac{dx}{(4-x^2)} = \frac{1}{4}\cdot\frac{dx}{(2^2-x^2)} = \frac{1}{4}\cdot\frac{1}{(2\times 2)}\log\left|\frac{2+x}{2-x}\right| + C = \frac{1}{16}\log\left|\frac{2+x}{2-x}\right| + C.$

21. Put $x^3 = t$ and $3x^2dx = dt$, we get

$$I = \frac{1}{3}\int \frac{dt}{(1-t^2)} = \frac{1}{3}\cdot\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C = \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C.$$

22. Put $x^2 = t$ and $2x\,dx = dt$.

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{(1-t^2)} = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C = \frac{1}{4}\log\left|\frac{1+x^2}{1-x^2}\right| + C.$$

23. Put $x^3 = t$ and $3x^2dx = dt$.

$$\therefore \quad I = \frac{1}{3}\int \frac{dt}{[(a^3)^2 - t^2]} = \frac{1}{3}\times\frac{1}{3a^3}\log\left|\frac{a^3+t}{a^3-t}\right| + C = \frac{1}{6a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right| + C.$$

24. $(3 - 2x - x^2) = 4 - (1 + 2x + x^2) = [2^2 - (1+x)^2]$.

$$\therefore \quad I = \int \frac{dx}{\{2^2-(1+x)^2\}} = \int \frac{dt}{(4-t^2)}, \text{ where } (1+x) = t$$

$$= \int \frac{dt}{(2^2-t^2)} = \frac{1}{2\times 2}\log\left|\frac{2+t}{2-t}\right| + C = \frac{1}{4}\log\left|\frac{2+(1+x)}{2-(1+x)}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{3+x}{1-x}\right| + C.$$

25. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(1-3\tan^2 x)}dx = \int \frac{dt}{(1-3t^2)}, \text{ where } \tan x = t$$

$$= \frac{1}{3}\int \frac{dt}{\left(\frac{1}{3}-t^2\right)} = \frac{1}{3}\cdot\int \frac{dt}{\left\{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2\right\}} = \frac{1}{3}\cdot\frac{1}{2\times\frac{1}{\sqrt{3}}}\log\left|\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right| + C$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C.$$

26. Put $\cot x = t$ and $-\operatorname{cosec}^2 x\, dx = dt$.

$$\therefore \quad I = -\int \frac{dt}{(1-t^2)} = -\frac{1}{(2\times 1)} \log\left|\frac{1+t}{1-t}\right| + C = \frac{-1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C.$$

27. $$I = \frac{1}{4}\int \frac{dx}{\left\{x^2 - \left(\frac{1}{2}\right)^2\right\}} = \frac{1}{4}\cdot\frac{1}{\left(2\times\frac{1}{2}\right)} \log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right| + C = \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C.$$

28. Put $x^2 = t$ and $2x\, dx = dt$.

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{(t^2-4^2)} = \frac{1}{2}\cdot\frac{1}{2\times 4}\log\left|\frac{t-4}{t+4}\right| = \frac{1}{16}\log\left|\frac{x^2-4}{x^2+4}\right| + C.$$

29. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(\tan^2 x - 4)} dx = \int \frac{dt}{(t^2-4)}, \text{ where } \tan x = t \text{ and } \sec^2 x\, dx = dt$$

$$= \int \frac{dt}{\{t^2 - 2^2\}} = \frac{1}{(2\times 2)}\log\left|\frac{t-2}{t+2}\right| + C = \frac{1}{4}\log\left|\frac{\tan x - 2}{\tan x + 2}\right| + C.$$

30. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(4\tan^2 x + 5)} dx = \int \frac{dt}{(4t^2+5)}, \text{ where } \tan x = t$$

$$= \frac{1}{4}\int \frac{dt}{\left(t^2 + \frac{5}{4}\right)} = \frac{1}{4}\int \frac{dt}{\left\{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right\}} = \frac{1}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)} \tan^{-1}\frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + C$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C.$$

31. $$I = \int \frac{\sin x}{(3\sin x - 4\sin^3 x)} dx = \frac{1}{(3-4\sin^2 x)} dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x} dx \text{ [on dividing Nr and Dr by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{(3-\tan^2 x)} dx = \int \frac{dt}{(3-t^2)}, \text{ where } \tan x = t$$

$$= \int \frac{dt}{\{(\sqrt{3})^2 - t^2\}} = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C.$$

32. On dividing Nr and Dr by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2 + 2\right\}} dx = \frac{dt}{t^2+2}, \text{ where } x - \frac{1}{x} = t$$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(x^2-1)}{\sqrt{2}x} + C.$$

OBJECTIVE QUESTIONS II

Mark (✓) against the correct answer in each of the following:

1. $\int \frac{dx}{\sqrt{4-9x^2}} = ?$

   (a) $\frac{1}{3}\sin^{-1}\frac{x}{3}+C$ (b) $\frac{2}{3}\sin^{-1}\left(\frac{2x}{3}\right)+C$

   (c) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$ (d) none of these

2. $\int \frac{dx}{\sqrt{16-4x^2}} = ?$

   (a) $\frac{1}{2}\sin^{-1}\frac{x}{2}+C$ (b) $\frac{1}{4}\sin^{-1}\frac{x}{2}+C$ (c) $\frac{1}{2}\sin^{-1}\frac{x}{4}+C$ (d) none of these

3. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} = ?$

   (a) $\sin^{-1}\frac{x}{2}+C$ (b) $\sin^{-1}\left(\frac{1}{2}\cos x\right)+C$

   (c) $\sin^{-1}(2\sin x)+C$ (d) $\sin^{-1}\left(\frac{1}{2}\sin x\right)+C$

4. $\int \frac{2^x}{\sqrt{1-4^x}}dx = ?$

   (a) $\sin^{-1}(2^x)\log 2+C$ (b) $\frac{\sin^{-1}(2^x)}{\log 2}+C$

   (c) $\sin^{-1}(2^x)+C$ (d) none of these

5. $\int \frac{dx}{\sqrt{2x-x^2}} = ?$

   (a) $\sin^{-1}(x+1)+C$ (b) $\sin^{-1}(x-2)+C$

   (c) $\sin^{-1}(x-1)+C$ (d) none of these

6. $\int \frac{dx}{\sqrt{x(1-2x)}} = ?$

   (a) $\frac{1}{\sqrt{2}}\sin^{-1}(2x-1)+C$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}(2x+1)+C$

   (c) $\frac{1}{\sqrt{2}}\sin^{-1}(4x+1)+C$ (d) $\frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+C$

7. $\int \frac{3x^2}{\sqrt{9-16x^6}}dx = ?$

(a) $\frac{1}{4}\sin^{-1}\left(\frac{x^3}{3}\right)+C$ (b) $\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+C$

(c) $4\sin^{-1}\left(\frac{x^3}{4}\right)+C$ (d) none of these

8. $\int \frac{dx}{\sqrt{2+2x-x^2}} = ?$

(a) $\sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C$ (b) $\sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

(c) $\sin^{-1}\sqrt{3}(x-1)+C$ (d) none of these

9. $\int \frac{dx}{\sqrt{16-6x-x^2}} = ?$

(a) $\sin^{-1}\left(\frac{x-3}{5}\right)+C$ (b) $\sin^{-1}\left(\frac{x+3}{5}\right)+C$

(c) $\frac{1}{5}\sin^{-1}(x+3)+C$ (d) none of these

10. $\int \frac{dx}{\sqrt{x-x^2}} = ?$

(a) $\sin^{-1}(x-1)+C$ (b) $\sin^{-1}(x+1)+C$

(c) $\sin^{-1}(2x-1)+C$ (d) none of these

11. $\int \frac{dx}{\sqrt{1+2x-3x^2}} = ?$

(a) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$

(c) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$ (d) none of these

12. $\int \frac{dx}{\sqrt{x^2-16}} = ?$

(a) $\sin^{-1}\left(\frac{x}{4}\right)+C$ (b) $\log\left|x+\sqrt{x^2-16}\right|+C$

(c) $\log\left|x-\sqrt{x^2-16}\right|+C$ (d) none of these

13. $\int \frac{dx}{\sqrt{4x^2-9}} = ?$

(a) $\frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right|+C$ (b) $\frac{1}{4}\log\left|x+\sqrt{4x^2-9}\right|+C$

(c) $\log\left|2x+\sqrt{4x^2-9}\right|+C$ (d) none of these

14. $\int \frac{x^2}{\sqrt{x^6-1}}dx = ?$

(a) $\frac{1}{2}\log\left|x^3+\sqrt{x^6-1}\right|+C$ (b) $\frac{1}{3}\log\left|x^3+\sqrt{x^6-1}\right|+C$

(c) $\frac{1}{3}\log\left|x^3-\sqrt{x^6-1}\right|+C$ (d) none of these

15. $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}} = ?$

(a) $-\frac{1}{2}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C$

(b) $-\frac{1}{3}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C$

(c) $-\frac{1}{6}\log\left|\cos x+\sqrt{2\cos^2 x-1}\right|+C$

(d) none of these

16. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}dx = ?$

(a) $\log\left|\tan x-\sqrt{\tan^2 x-4}\right|+C$ (b) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(c) $\frac{1}{2}\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$ (d) none of these

17. $\int \frac{dx}{(1-e^{2x})} = ?$

(a) $\log\left|e^x+\sqrt{e^{2x}-1}\right|+C$ (b) $\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$

(c) $-\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$ (d) none of these

18. $\int \frac{dx}{\sqrt{x^2-3x+2}} = ?$

(a) $\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$ (b) $\log\left|x+\sqrt{x^2-3x+2}\right|+C$

(c) $\log\left|x-\sqrt{x^2-3x+2}\right|+C$ (d) none of these

19. $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx = ?$

(a) $\log \left| \sin x + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(b) $\log \left| (\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(c) $\log \left| (\sin x - 1) - \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(d) none of these

20. $\int \frac{dx}{\sqrt{2 - 4x + x^2}} = ?$

(a) $\log \left| (x-2) + \sqrt{x^2 - 4x + 2} \right| + C$ (b) $\log \left| x + \sqrt{x^2 - 4x + 2} \right| + C$

(c) $\log \left| x - \sqrt{x^2 - 4x + 2} \right| + C$ (d) none of these

21. $\int \frac{dx}{\sqrt{x^2 + 6x + 5}} = ?$

(a) $\log \left| x + \sqrt{x^2 + 6x + 5} \right| + C$ (b) $\log \left| x - \sqrt{x^2 + 6x + 5} \right| + C$

(c) $\log \left| (x+3) + \sqrt{x^2 + 6x + 5} \right| + C$ (d) none of these

22. $\int \frac{dx}{\sqrt{(x-3)^2 - 1}} = ?$

(a) $\log \left| (x-3) + \sqrt{x^2 - 6x + 8} \right| + C$ (b) $\log \left| x + \sqrt{x^2 - 6x + 8} \right| + C$

(c) $\log \left| (x-3) - \sqrt{x^2 - 6x + 8} \right| + C$ (d) none of these

23. $\int \frac{dx}{\sqrt{x^2 - 6x + 10}} = ?$

(a) $\log \left| x + \sqrt{x^2 - 6x + 10} \right| + C$ (b) $\log \left| (x-3) + \sqrt{x^2 - 6x + 10} \right| + C$

(c) $\log \left| x - \sqrt{x^2 - 6x + 10} \right| + C$ (d) none of these

24. $\int \frac{x^2 dx}{\sqrt{x^6 + a^6}} dx = ?$

(a) $\frac{1}{3} \log |x^6 + a^6| + C$ (b) $\frac{1}{3} \tan^{-1} \left(\frac{x^3}{a^3} \right) + C$

(c) $\frac{1}{3} \log \left| x^3 + \sqrt{x^6 + a^6} \right| + C$ (d) none of these

25. $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}dx = ?$

(a) $\log\left|\tan x+\sqrt{\tan^2 x+16}\right|+C$ (b) $\log\left|x+\sqrt{\tan^2 x+16}\right|+C$

(c) $\log\left|\tan x-\sqrt{\tan^2 x+16}\right|+C$ (d) none of these

26. $\int \frac{dx}{\sqrt{3x^2+6x+12}} = ?$

(a) $\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$ (b) $\frac{1}{3}\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$

(c) $\frac{1}{\sqrt{3}}\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$ (d) none of these

27. $\int \frac{dx}{\sqrt{2x^2+4x+6}} = ?$

(a) $\frac{1}{2}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

(b) $\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

(c) $\frac{1}{\sqrt{2}}\log\left|x+\sqrt{x^2+2x+3}\right|+C$

(d) none of these

28. $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}dx = ?$

(a) $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+C$

(b) $\log\left|x^3+\sqrt{x^6+2x^3+3}\right|+C$

(c) $\frac{1}{3}\log\left|(x^3+1)-\sqrt{x^6+2x^3+3}\right|+C$

(d) none of these

29. $\int \sqrt{4-x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$ (b) $x\sqrt{4-x^2}+\sin^{-1}\frac{x}{2}+C$

(c) $\frac{1}{2}x\sqrt{4-x^2}-2\sin^{-1}\frac{x}{2}+C$ (d) none of these

30. $\int \sqrt{1-9x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{1-9x^2}+\frac{1}{18}\sin^{-1}3x+C$ (b) $\frac{3x}{2}\sqrt{1-9x^2}+\frac{1}{6}\sin^{-1}3x+C$

(c) $\frac{x}{2}\sqrt{1-9x^2}+\frac{1}{6}\sin^{-1}3x+C$ (d) none of these

31. $\int \sqrt{9-4x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{9-4x^2} + \frac{9}{4}\sin^{-1}\frac{2x}{3} + C$ (b) $x\sqrt{9-4x^2} + \frac{9}{2}\sin^{-1}\frac{2x}{3} + C$

(c) $\frac{x}{2}\sqrt{9-4x^2} - \frac{9}{4}\sin^{-1}\frac{2x}{3} + C$ (d) none of these

32. $\int \cos x\sqrt{9-\sin^2 x}\,dx = ?$

(a) $\frac{1}{2}\sin x\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(b) $\frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(c) $\frac{1}{2}\cos x\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(d) none of these

33. $\int \sqrt{x^2-16}\,dx = ?$

(a) $x\sqrt{x^2-16} - 4\log\left|x+\sqrt{x^2-16}\right| + C$

(b) $\frac{x}{2}\sqrt{x^2-16} - 8\log\left|x+\sqrt{x^2-16}\right| + C$

(c) $\frac{x}{2}\sqrt{x^2-16} + 8\log\left|x+\sqrt{x^2-16}\right| + C$

(d) none of these

34. $\int \sqrt{x^2-4x+2}\,dx = ?$

(a) $\frac{1}{2}(x-2)\sqrt{x^2-4x+2} + \log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(b) $(x-2)\sqrt{x^2-4x+2} + \frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(c) $\frac{1}{2}(x-2)\sqrt{x^2-4x+2} - \log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(d) none of these

35. $\int \sqrt{9x^2+16}\,dx = ?$

(a) $\frac{x}{2}\sqrt{9x^2+16} + \frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right| + C$

(b) $\frac{x}{2}\sqrt{9x^2+16} - \frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right| + C$

(c) $x\sqrt{9x^2+16} + 24\log\left|3x+\sqrt{9x^2+16}\right| + C$

(d) none of these

36. $\int e^x \sqrt{e^{2x}+4}\,dx = ?$

(a) $\frac{1}{2}e^x\sqrt{e^{2x}+4} - 2\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(b) $\frac{1}{2}e^x\sqrt{e^{2x}+4} + 2\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(c) $e^x\sqrt{e^{2x}+4} + \frac{1}{2}\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(d) none of these

37. $\int \frac{\sqrt{16+(\log x)^2}}{x}dx = ?$

(a) $\frac{1}{2}\log x\cdot\sqrt{16+(\log x)^2} + 8\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(b) $\frac{1}{2}\log x\cdot\sqrt{16+(\log x)^2} + 4\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(c) $\log x\cdot\sqrt{16+(\log x)^2} + 16\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(d) none of these

ANSWERS (OBJECTIVE QUESTIONS II)

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (d)	**7.** (b)	**8** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (a)
21. (c)	**22.** (a)	**23.** (b)	**24.** (c)	**25.** (a)	**26.** (c)	**27.** (b)	**28.** (a)	**29.** (a)	**30.** (c)
31. (a)	**32.** (a)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (a)			

HINTS TO THE GIVEN OBJECTIVE QUESTIONS II

1. $I = \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\cdot\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}$

$= \frac{1}{3}\sin^{-1}\left(\frac{x}{2/3}\right) + C.$

2. $I = \int \frac{dx}{\sqrt{16-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2^2-x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{2} + C.$

3. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\frac{t}{2} + C = \sin^{-1}\left(\frac{1}{2}\sin x\right) + C.$$

4. Put $2^x = t$ and $2^x \log 2\,dx = dt$.

$$\therefore \qquad I = \frac{1}{\log 2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\log 2}\sin^{-1}t + C = \frac{1}{(\log 2)}\sin^{-1}(2^x) + C.$$

5. $\sqrt{2x-x^2}=\sqrt{1-(1-2x+x^2)}=\sqrt{1-(x-1)^2}$

$\therefore \quad I=\int\frac{dx}{\sqrt{1-(x-1)^2}}=\int\frac{dt}{\sqrt{1-t^2}}$, where $(x-1)=t$

$=\sin^{-1}t+C=\sin^{-1}(x-1)+C.$

6. $x(1-2x)=(x-2x^2)=2\left(\frac{x}{2}-x^2\right)=2\left[\frac{1}{16}-\left(x^2-\frac{x}{2}+\frac{1}{16}\right)\right]=2\left[\left(\frac{1}{4}\right)^2-\left(x-\frac{1}{4}\right)^2\right]$

$\therefore \quad I=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}}=\frac{1}{\sqrt{2}}\cdot\int\frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2-t^2}}$, where $\left(x-\frac{1}{4}\right)=t$

$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{t}{(1/4)}+C=\frac{1}{\sqrt{2}}\sin^{-1}4t+C=\frac{1}{\sqrt{2}}\sin^{-1}\left\{4\left(x-\frac{1}{4}\right)\right\}+C$

$=\frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+C.$

7. Put $x^3=t$ and $3x^2dx=dt.$

$\therefore \quad I=\int\frac{dt}{\sqrt{9-16t^2}}=\frac{1}{4}\int\frac{dt}{\sqrt{\frac{9}{16}-t^2}}=\frac{1}{4}\cdot\int\frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-t^2}}$

$=\frac{1}{4}\sin^{-1}\frac{t}{(3/4)}+C=\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+C.$

8. $(2+2x-x^2)=3-(1+x^2-2x)=(\sqrt{3})^2-(x-1)^2$

$\therefore \quad I=\int\frac{dx}{\sqrt{(\sqrt{3})^2-(x-1)^2}}=\int\frac{dt}{\sqrt{(\sqrt{3})^2-t^2}}$, where $(x-1)=t$ and $dx=dt$

$=\sin^{-1}\frac{t}{\sqrt{3}}+C=\sin^{-1}\frac{(x-1)}{\sqrt{3}}+C.$

9. $(16-6x-x^2)=[-(x^2+6x+9)+25]=\{25-(x+3)^2\}$

$\therefore \quad I=\int\frac{dx}{\sqrt{5^2-(x+3)^2}}=\int\frac{dt}{\sqrt{5^2-t^2}}=\sin^{-1}\frac{t}{5}+C=\sin^{-1}\left(\frac{x+3}{5}\right)+C.$

10. $(x-x^2)=\frac{1}{4}-\left(x^2-x+\frac{1}{4}\right)=\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2$

$\therefore \quad I=\int\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}=\int\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-t^2}}=\sin^{-1}\frac{t}{(1/2)}+C$

$=\sin^{-1}2t+C=\sin^{-1}2\left(x-\frac{1}{2}\right)+C=\sin^{-1}(2x-1)+C.$

11. $(1+2x-3x^2)=-3\left(x^2-\frac{2}{3}x-\frac{1}{3}\right)=3\left\{\frac{4}{9}-\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)\right\}$

$=3\left\{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2\right\}.$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}} = \frac{1}{\sqrt{3}} \cdot \int \frac{dt}{\sqrt{\left(\frac{2}{3}\right)^2 - t^2}} = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{t}{\frac{2}{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3t}{2}\right) + C = \frac{1}{\sqrt{3}} \sin^{-1} \frac{(3x-1)}{2} + C.$$

12. $I = \int \frac{dx}{\sqrt{x^2 - 4^2}} = \log\left|x + \sqrt{x^2 - 16}\right| + C.$

13. $I = \frac{1}{2}\int \frac{dx}{\sqrt{x^2 - \left(\frac{3}{2}\right)^2}} = \frac{1}{2}\log\left|x + \sqrt{x^2 - \frac{9}{4}}\right| + C = \frac{1}{2}\log\left|2x + \sqrt{4x^2 - 9}\right| + C.$

14. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 - 1}} = \frac{1}{3}\log\left|t + \sqrt{t^2 - 1}\right| + C = \frac{1}{3}\log\left|x^3 + \sqrt{x^6 - 1}\right| + C.$$

15. Put $\cos x = t$ and $\sin x\, dx = -dt$. Then,

$$I = -\int \frac{dt}{\sqrt{4t^2 - 1}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 - \frac{1}{4}}} = -\frac{1}{2}\log\left|t + \sqrt{t^2 - \frac{1}{4}}\right| + C$$

$$= -\frac{1}{2}\log\left|2t + \sqrt{4t^2 - 1}\right| + C = -\frac{1}{2}\log\left|2\cos x + \sqrt{4\cos^2 x - 1}\right| + C.$$

16. Put $\tan x = t$ and $\sec^2 x\, dx = dt$. Then,

$$I = \int \frac{dt}{\sqrt{t^2 - 2^2}} = \log\left|t + \sqrt{t^2 - 4}\right| + C = \log\left|\tan x + \sqrt{\tan^2 x - 4}\right| + C.$$

17. $I = \int \frac{e^{-x}dx}{\sqrt{e^{-2x}(1 - e^{2x})}} = \int \frac{e^{-x}}{\sqrt{e^{-2x} - 1}}dx = -\int \frac{dt}{\sqrt{(t^2 - 1)}}$, where $e^{-x} = t$

$$= -\log\left|t + \sqrt{t^2 - 1}\right| + C = -\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + C.$$

18. $I = \int \frac{dx}{\sqrt{\left(x^2 - 3x + \frac{9}{4}\right) - \frac{1}{4}}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$$= \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} = \log\left|t + \sqrt{t^2 - \frac{1}{4}}\right| + C, \text{ where } \left(x - \frac{3}{2}\right) = t$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C.$$

19. Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{(t^2 - 2t + 1) - 4}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

$$= \log\left|(t-1) + \sqrt{(t-1)^2 - 2^2}\right| + C = \log\left|(t-1) + \sqrt{t^2 - 2t - 3}\right| + C$$

$$= \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}\right| + C.$$

20. $I = \int \frac{dx}{\sqrt{(x^2 - 4x + 4) - 2}} = \int \frac{dx}{\sqrt{(x-2)^2 - (\sqrt{2})^2}}$

$= \log\left|(x-2) + \sqrt{x^2 - 4x + 2}\right| + C.$

21. $I = \int \frac{dx}{\sqrt{(x^2 + 6x + 9) - 4}} = \int \frac{dx}{\sqrt{(x+3)^2 - 2^2}} = \log\left|(x+3) + \sqrt{x^2 + 6x + 5}\right| + C.$

22. $I = \log\left|(x-3) + \sqrt{x^2 - 6x + 8}\right| + C.$

23. $I = \int \frac{dx}{\sqrt{(x-3)^2 + 1^2}} = \log\left|(x-3) + \sqrt{x^2 - 6x + 10}\right| + C.$

24. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + (a^3)^2}} = \frac{1}{3}\log\left|t + \sqrt{t^2 + a^6}\right| + C = \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C.$$

25. Put $\tan x = t$ and $\sec^2 x\, dx = dt$. Then,

$$I = \int \frac{dt}{\sqrt{t^2 + 4^2}} = \log\left|t + \sqrt{t^2 + 16}\right| = \log\left|\tan x + \sqrt{\tan^2 x + 16}\right| + C.$$

26. $I = \int \frac{dx}{\sqrt{3(x^2 + 2x + 4)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2 + 2x + 4}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{3})^2}}$

$= \frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2 + (\sqrt{3})^2}} = \frac{1}{\sqrt{3}}\log\left|t + \sqrt{t^2 + 3}\right| + C$

$= \frac{1}{\sqrt{3}} \cdot \log\left|(x+1) + \sqrt{x^2 + 2x + 4}\right| + C.$

27. $I = \int \frac{dx}{\sqrt{2(x^2 + 2x + 3)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2 + 2x + 3}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$

$= \frac{1}{\sqrt{2}}\log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C.$

28. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + 2t + 3}} = \frac{1}{3}\int \frac{dt}{\sqrt{(t+1)^2 + (\sqrt{2})^2}}$$

$$= \frac{1}{3}\log\left|(t+1) + \sqrt{t^2 + 2t + 3}\right| + C$$

$$= \frac{1}{3}\log\left|(x^3 + 1) + \sqrt{x^6 + 2x^3 + 3}\right| + C.$$

29. Using $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$,

$$\int \sqrt{4 - x^2}\, dx = \int \sqrt{2^2 - x^2}\, dx = \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C.$$

30. $\int \sqrt{1 - 9x^2}\, dx = 3\int \sqrt{\frac{1}{9} - x^2}\, dx = 3\left[\frac{x}{2}\sqrt{\frac{1}{9} - x^2} + \frac{1}{18}\sin^{-1}\frac{x}{(1/3)}\right] + C$

$$= \frac{3x}{2}\sqrt{\frac{1}{9} - x^2} = \frac{1}{6}\sin^{-1} 3x + C = \frac{x}{2}\sqrt{1 - 9x^2} + \frac{1}{6}\sin^{-1} 3x + C.$$

31. $\int \sqrt{9-4x^2}\,dx = 2\int \sqrt{\frac{9}{4}-x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\frac{x}{(3/2)}\right] + C$

$= \frac{x}{2}\sqrt{9-4x^2} + \frac{9}{4}\sin^{-1}\frac{2x}{3} + C.$

32. Put $\sin x = t$ and $\cos x\,dx = dt$. Then,

$I = \int \sqrt{9-t^2}\,dt = \frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\frac{t}{3} + C$

$= \frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C.$

33. $\int \sqrt{x^2-16}\,dx = \int \sqrt{x^2-4^2}\,dx = \frac{x}{2}\sqrt{x^2-4^2} - \frac{16}{2}\log\left|x+\sqrt{x^2-16}\right| + C$

$= \frac{x}{2}\sqrt{x^2-16} - 8\cdot\log\left|x+\sqrt{x^2-16}\right| + C.$

34. $I = \int \sqrt{(x-2)^2-(\sqrt{2})^2}\,dx$

$= \frac{1}{2}(x-2)\sqrt{x^2-4x+2} - \frac{2}{2}\log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

$= \frac{1}{2}(x-2)\sqrt{x^2-4x+2} - \log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C.$

35. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$\therefore \quad I = 3\int \sqrt{x^2+\frac{16}{9}}\,dx = 3\int \sqrt{x^2+\left(\frac{4}{3}\right)^2}\,dx$

$= 3\left\{\frac{x}{2}\sqrt{x^2+\frac{16}{9}} + \frac{8}{9}\log\left|x+\sqrt{x^2+\frac{16}{9}}\right|\right\} + C$

$= \frac{x}{2}\sqrt{9x^2+16} + \frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right| + C.$

36. Put $e^x = t$ and $e^x dx = dt$. Then,

$I = \int \sqrt{t^2+4}\,dt = \int \sqrt{t^2+2^2}\,dt = \frac{t}{2}\sqrt{t^2+4} + \frac{4}{2}\log\left|t+\sqrt{t^2+4}\right| + C$

$= \frac{1}{2}e^x\sqrt{e^{2x}+4} + 2\log\left|e^x+\sqrt{e^{2x}+4}\right| + C.$

37. Put $\log x = t$ and $\frac{1}{x}dx = dt$. Then,

$I = \int \sqrt{t^2+(4)^2} = \frac{t}{2}\sqrt{t^2+16} + \frac{16}{2}\log\left|t+\sqrt{t^2+16}\right| + C$

$= \frac{1}{2}\log x\sqrt{(\log x)^2+16} + 8\log\left|\log x+\sqrt{(\log x)^2+16}\right| + C.$